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Rigidity for Einstein Manifolds under Bounded Covering Geometry

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In honor of Professor Xiaochun Rong on his seventieth birthday

Abstract. In this note, we prove three rigidity results for Einstein manifolds with bounded covering geometry. (1) An almost flat manifold (M,g) must be flat if it is Einstein, i.e. $\mathrm{Ric}_g=\lambda g$ for some real number λ. (2) A compact Einstein manifold with a non-vanishing and almost maximal volume entropy is hyperbolic. (3) A compact Einstein manifold admitting a uniform local rewinding almost maximal volume is isometric to a space form.

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1 Introduction

Several years ago, Rong proposed a program that aims to study the geometry and topology of manifolds with lower bounded Ricci curvature and local bounded covering geometry [29], see the survey paper [30]. Following the program, we prove three rigidity results for Einstein manifolds under local bounded covering geometry.

The first one is that an almost flat manifold must be flat if it is Einstein. Let us recall that Gromov's theorem [21,42] on almost flat manifolds says that for any positive integer $n>0$, there are $\epsilon(n),C(n)>0$ such that for any compact n-manifold (M,g), if the rescaling invariant

$$\mathrm{diam}(M,g)^2\cdot\max|\mathrm{Sec}_g|<\epsilon(n),$$

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then M is diffeomorphic to a infra-nilmanifold N/Γ, where N is a simply connected nilpotent group, and Γ is a subgroup of affine group $N \rtimes \mathrm{Aut}(N)$ of N such that $[\Gamma:\Gamma\cap N] \leq C(n)$.

There are many compact Ricci-flat manifolds including complex $K3$ surfaces and $G2$ manifolds. Though it generally fails, Gromov's theorem, however, still holds at the level of fundamental group of manifolds under lower bounded Ricci curvature. That is, there is $\epsilon(n), C(n) > 0$ such that for any compact n-manifold (M,g) of almost non-negative Ricci curvature, i.e.,

$$\mathrm{diam}(M,g)^2 \cdot \mathrm{Ric}_g > -\epsilon(n),$$

its fundamental group $\pi_1(M)$ is virtually $C(n)$-nilpotent, i.e. $\pi_1(M)$ contains a nilpotent subgroup N with index $[\pi_1(M):N] \leq C(n)$. It was originally conjectured by Gromov [22], and proved by Kapovitch-Wilking [31], which now is called the generalized Margulis lemma. In fact, the polycyclic rank of $\pi_1(M)$ is well-defined and is no more than n, which is that of any finite-indexed nilpotent subgroup N ([38, §2.4]), i.e., the number of $\mathbb{Z}$ appears in a polycyclic series as cyclic factors.

Recently a criterion for almost flat manifold theorem under lower bounded Ricci curvature was proved by Huang-Kong-Rong-Xu [29]. Combining with Naber-Zhang [38], we have the following result.

Theorem 1.1 (Almost flat theorem under lower bounded Ricci curvature, [29,38])**.** *There is $\epsilon(n) > 0, v(n) > 0$ such that for any n-manifold (M,g) of Ricci curvature $\geq -(n-1)$ and $\mathrm{diam}(M,g) < \epsilon(n)$, the followings are equivalent:*

(1) M is diffeomorphic to a infra-nil manifold;

(2) the polycyclic rank of $\pi_1(M)$ is equal to n;

(3) (M,g) satisfies $(1,v(n))$-bounded covering geometry, i.e. $\mathrm{vol}\,\underbrace{B_1(\tilde{x})} \geq v(n) > 0$, where $\tilde{x}$ is a preimage point of x in the Riemannian universal cover $\widetilde{M} = \widetilde{B_1(x)}$.

In the above theorem, it is well-known by the structure of nilpotent Lie group that the polycyclic rank of $N \cap \pi_1(M)$ is equal to n. Hence, (1) trivially implies (2). Conversely, it was proved by Kapovitch-Wilking [31] that (2) implies M is homeomorphic to an infra-nilmanifold. Later, it was proved in [29, Theorem A] that (3) implies (1), and by [38, Proposition 5.9], (2) implies (3).

Theorem 1.1 can be viewed as a natural extension of Colding's maximal Betti number theorem [18], i.e, under the condition of Theorem 1.1, M is diffeomorphic to a n-torus if and only if its first Betti number equals n.

In this note, we prove that if in addition (M,g) is Einstein in Theorem 1.1, then (M,g) must be flat.

Theorem 1.2 (Rigidity for almost nonnegative Ricci curvature)**.** *There is $\epsilon(n) > 0, v(n) > 0$ such that for any Einstein n-manifold (M,g) with $\mathrm{Ric}_g = \lambda g$ and $\mathrm{diam}(M,g)^2 \cdot \lambda > -\epsilon(n)$, the followings are equivalent:*