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An Alternative Proof for the Upper Bound of Curvature Integral on Manifolds with Lower Sectional Curvature Bound

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In honor of Professor Xiaochun Rong on his seventieth birthday

Abstract. Petrunin proved that the integral of scalar curvature in a unit ball is bounded from above in terms of the dimension of the manifold and the lower bound of the sectional curvature. In this paper, we give an alternative proof for this result. The main difference between this proof and Petrunin's original proof is that we construct a stratified finite covering and apply it directly to the given manifold, rather than arguing by contradiction for a sequence of manifolds, which satisfy some technical lifting properties.

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Key words: Scalar curvature, sectional curvature, L^1-norm, Alexandrov space, stratification, covering.

1 Introduction

The following question was raised by Yau in 1992 [9].

Question 1.1 (Problem 9, [9]). Let (M,g) be an n-dimensional manifolds with non-negative Ricci curvature and $p\in M$. Does it holds that

$$\lim_{R\to\infty} R^{-(n-2)}\int_{B_R(p)} scal\,\mathrm{d}vol_g<\infty? \tag{1.1}$$

As a stronger version of the above statement, it was conjectured that

$$\int_{B_1(p)} scal\,\mathrm{d}vol_g<C(n), \tag{1.2}$$

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provided $\mathrm{Ric}_{B_2(p)} \geq -1$. This indicates that if the Ricci curvature is bounded from below, then the region where the scalar curvature is large doesn't have large volume. One can also compare it to the Gauss-Bonnet Theorem in dimension 2. Despite some progress (see the following theorem) in proving the conjecture under stronger conditions, it still remains open.