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Moduli Spaces of Metrics with Lower or Pinched Curvature Bounds and Their Compactifications

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In honor of Professor Xiaochun Rong on his seventieth birthday

Abstract. We present a metric approach to the study of moduli spaces of metrics with certain curvature bounds and suitable other geometric constraints, and their compactifications. This is accompanied by a deeper discussion of related results, questions and problems in the realm of positive and positively pinched sectional as well as Ricci curvature. Regarding the latter two topics, we place special emphasis on corresponding works and contributions of Rong X and his collaborators.

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1 Introduction

A central theme of global Riemannian geometry consists in the construction of Riemannian metrics with certain geometric constraints like, say, curvature bounds, on a given smooth manifold M. However, once the existence problem has been solved, an equally important task is then to study the question what the space of all such metrics on M does look like. Moreover, one also seeks to understand the structure of the corresponding moduli space of these metrics. The latter is defined as the quotient of the former by the diffeomorphism group of the manifold, acting on the space of metrics or suitable subspaces by pulling back metrics.

These spaces are customarily equipped with the topology of smooth convergence on compact subsets and the corresponding quotient topology, respectively. Their topological properties then provide the right means to measure 'how many' different metrics, or geometries, respectively, the given manifold actually does exhibit. For further background and details, compare the monograph [31].

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In the present note, we first propose and use a different method to study moduli spaces of Riemannian metrics. For simplicity, let us assume that the manifold M in question is compact. We can then view a moduli space $\mathcal{M}(M)$ of metrics on M in the former sense also as a subspace of the Gromov-Hausdorff space $(\mathcal{M},d_{GH})$ of isometry classes of compact metric spaces, equipped with the Gromov-Hausdorff distance. Indeed, since smooth (or C^k or $C^{k,\alpha}$) convergence implies Gromov-Hausdorff convergence, one can think of $\mathcal{M}(M)$ as being continuously embedded into $(\mathcal{M},d_{GH})$. † Moreover, this point of view also allows to obtain and study natural subspace compactifications of $\mathcal{M}(M)$.

In this regard, we shall discuss below several conditions based on Gromov's celebrated Precompactness Theorem. In addition, in this setting also powerful methods and theorems from the theory of convergence and collapsing of Riemannian manifolds can be employed.

We will then examine in more detail related results and questions in the realm of positive sectional or Ricci curvature, especially under pinching conditions. Here, we focus on ones that appear in or arise from the work of Rong X and his collaborators.

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2 A metric approach to moduli spaces of Riemannian metrics

To approach the investigation of isometry classes of metrics and their moduli spaces' potential compactifications via Gromov-Hausdorff distance, let us first discuss some curvature and other geometric conditions that will ensure precompactness with respect to those, so that we will always be able to work with and compare to limits‡. As already mentioned, for simplicity we will only consider closed manifolds, but notice again that everything described in this present context will allow for obvious pointed and/or equivariant generalisations.

For a closed smooth manifold M, let $\mathcal{ISO}(M,\mathcal{C},L)$ denote the set of isometry classes of Riemannian metrics g on M which satisfy for a given set of geometric constraints $\mathcal{C}$ and a given nonnegative real number L an inequality of the form $\mathcal{C}(g) \leq L$. Here we want our conditions $\mathcal{C}$ to be chosen in such a way that

(1) $\mathcal{ISO}(M,\mathcal{C},L)$ is precompact with respect to the Gromov-Hausdorff distance, and such that

(2) for every Riemannian metric g on M there exists some real number L with $g \in \mathcal{ISO}(M,\mathcal{C},L)$.

†Using pointed and/or equivariant Gromov-Hausdorff convergence, the same reasoning does, of course, also apply to non-compact and/or equivariant settings.

‡The roots of what follows actually date back a long time ago and emerged from inspiring discussions with Anton Petrunin. Spasibo, Tosha!