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On Conical Asymptotically Flat Manifolds

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Abstract. We prove a conjecture of Petrunin and Tuschmann on the non-existence of asymptotically flat 4-manifolds asymptotic to the half plane. We also survey recent progress and questions concerning gravitational instantons, which serve as our motivation for studying this question.

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1 Introduction

In this paper we study conical asymptotically flat ($\mathcal{AF}$) manifolds. By definition, a complete non-compact m dimensional Riemannian manifold (M^m,g,p) is called

- $\mathcal{AF}$ if the Riemannian curvature decays at the rate

$$|Rm_g| = o(r^{-2}), \quad r \to \infty,$$

  where r denotes the distance to p; in other words, the *asymptotic curvature*

$$A(M,g) \equiv \limsup_{r\to\infty} r^2 |Rm_g|$$

  is zero.

- *conical* if it is asymptotic to a unique metric cone $\mathcal{C}_\infty$ at infinity, i.e., $(M,\lambda^{-2}g,p)$ converges in the pointed Gromov-Hausdorff topology to $\mathcal{C}_\infty$ as $\lambda \to \infty$. Here the *asymptotic cone* $\mathcal{C}_\infty$ may have a lower dimension.

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We need to make a few remarks about the terminologies here. First the words "asymptotically flat" may refer to properties of varying generality in the literature. The notion of $\mathcal{AF}$ we use here is the same as the notion of asymptotically flat used in [34]. A related notion is what we shall call *strongly* $\mathcal{AF}$, which means that $|Rm_g|$ is bounded by a positive decreasing function $f(r)$ with

$$\int_0^\infty rf(r)dr<\infty.$$

An even stronger condition is that of *faster than quadratic curvature decay*, which means that

$$|Rm_g|\leqslant r^{-2-\epsilon}$$

for some $\epsilon>0$ as $r\to\infty$. We also try to distinguish the notation $\mathcal{AF}$ from AF, which usually refers to being asymptotic to a specific flat model end (see Section 3).

There has been extensive work in Riemannian geometry studying topological and geometrical properties of $\mathcal{AF}$ manifolds. Recall that flat ends of Riemannian manifolds were completely classified by Eischenburg-Schroeder [17]. Ends of $\mathcal{AF}$ manifolds are much more flexible; for example, any 2 dimensional non-compact surface admits an $\mathcal{AF}$ metric [1]. It is known [22, 30] that strongly $\mathcal{AF}$ manifolds are automatically conical. However there are interesting examples of conical $\mathcal{AF}$ manifolds which are not strongly $\mathcal{AF}$. Examples include the family of ALG* hyperkähler metrics [20].

Our interest in $\mathcal{AF}$ manifolds is motivated by the fact that they provide potential model ends for complete Ricci-flat metrics. In 4 dimensions a complete non-compact Ricci-flat manifold with

$$\int |Rm_g|^2<\infty$$

is called a *gravitational instanton*. The readers should be warned that there are varying definitions of gravitational instantons in the literature. Some require the extra assumption of being hyperkähler, but we do not make this hypothesis in the current paper. All known examples of gravitational instantons are conical and they are all $\mathcal{AF}$ except the family of ALH* hyperkähler metrics ([20, 38]) and the family constructed in [23] (which has fundamental group $\mathbb{Z}$). The latter are known to have *nilpotent* asymptotic geometries.

For general conical $\mathcal{AF}$ manifolds, in 2001, Petrunin and Tuschmann [34] proved a structural result regarding the end structure. In particular, they proved that there are only finitely many ends and a simply connected end must have asymptotic cone the flat $\mathbb{R}^m$ if $m\neq 4$ and the flat $\mathbb{R}^4$,$\mathbb{R}^3$ or the half plane

$$\mathbb{H}\equiv\mathbb{R}_{\geqslant 0}\times\mathbb{R}$$

when $m=4$. It follows that when $m\neq 4$ the metric has Euclidean volume growth and there is no *collapsing* phenomena along the convergence to the asymptotic cone. But when $m=4$ collapsing may indeed occur so the situation is more interesting. Clearly $\mathbb{R}^4$ can be realized. Also $\mathbb{R}^3$ can be realized as the asymptotic cone of the Taub-NUT gravitational