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Pair-of-Pants Decompositions of 4-Manifolds Diffeomorphic to General Type Hypersurfaces

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In honor of Professor Xiaochun Rong on his seventieth birthday

Abstract. In this paper, we show that a smooth 4-manifold diffeomorphic to a complex hypersurface in $\mathbb{CP}^3$ of degree $d \geq 5$ can be decomposed as the union of $d(d-4)^2$ copies of 4-dimensional pair-of-pants and certain subsets of K3 surfaces.

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Key words: Pair-of-pants decompositions, general type hypersurfaces.

1 Introduction

A compact Riemann surface Σ admits Riemannian metrics of constant Gaussian curvature, while a 3-dimensional manifold may not admit any constant curvature metric. Instead the Perelman's theorem on the Thurston's geometrisation conjecture asserts that a 3-manifold can be canonically decomposed into domains, and some of them carry complete constant curvature Riemannian metrics of finite volume ([29, 44]). A difference is that unlike the 3-dimensional case, if the genus of a Riemann surface is bigger than one, the hyperbolic metrics are not unique, and form a moduli space. However, Σ admits so called pair-of-pants decompositions which restore certain aspects of the uniqueness (It is a standard topic in topology. See 'Pair of pants (mathematics)' in www.wikipedia.org).

A pair-of-pants $\mathcal{P}^1$ of real dimension two, or complex dimension one, is defined as the complement set of three generic points in $\mathbb{CP}^1$, i.e. $\mathcal{P}^1 = \mathbb{CP}^1 \backslash \{0,1,\infty\}$. There is a unique complete Riemannian metric g on $\mathcal{P}^1$ with Gaussian curvature -1 and finite volume $\mathrm{Vol}_g(\mathcal{P}^1) = 2\pi$. There is a fibration structure $\mathcal{P}^1 \to \mathrm{Y}$ from $\mathcal{P}^1$ to a graph of Y-shape with generic fibres S^1, and one singular fibre of shape $\ominus$ over the vertex.

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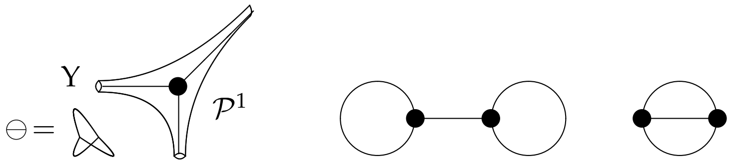

Figure 1: Fibration $\mathcal{P}^1 \to \mathrm{Y}$, and two graphs associated with pair-of-pants decompositions of Riemann surfaces of genus 2.

If Σ is a compact Riemann surface of genus $g \geq 2$, then there is an open subset $\Sigma^o \subset \Sigma$ consisting exactly of $2g-2$ copies of pair-of-pants, and the complement $\Sigma \backslash \Sigma^o$ is the disjoint union of $3g-3$ circles. Each circle is not homotopic to a constant curve in Σ. Therefore, Σ is decomposed into canonical local pieces, the pair-of-pants, glued along circles. The Gauss-Bonnet formula expresses the Euler number via the volume of g

$$\chi(\Sigma) = -\frac{1}{2\pi} \sum^{2g-2} \mathrm{Vol}_{\mathrm{g}}(\mathcal{P}^1) = \frac{1-g}{\pi} \mathrm{Vol}_{\mathrm{g}}(\mathcal{P}^1). \tag{1.1}$$

The combinatoric structure of the decomposition $\Sigma = \Sigma^o \cup (\Sigma \backslash \Sigma^o)$ is represented by a cubic graph, i.e., any vertex is of shape Y, where each pair-of-pants corresponds to one vertex, and any edge associates with a circle in $\Sigma \backslash \Sigma^o$.

In [38], Mikhalkin has generalised the notion of pair-of-pants to the case of any even dimension, and proved that a smooth complex hypersurface in $\mathbb{CP}^{n+1}$, and more general toric manifolds, admits pair-of-pants decompositions. [24] studied pair-of-pants decompositions for real 4-dimensional manifolds from the topology perspective. It is shown in [24] that every finitely presented group is the fundamental group of a 4-manifold admitting pair-of-pants decompositions. In this paper, we study some differential/algebraic geometry aspects of pair-of-pants decompositions for general type hypersurfaces in $\mathbb{CP}^3$.

A pair-of-pants $\mathcal{P}^2$ of real dimension 4, equivalently complex dimension 2, is defined as the complement of 4 general positioned lines D in $\mathbb{CP}^2$, i.e. $\mathcal{P}^2 = \mathbb{CP}^2 \backslash D$ ([38]), where D can be chosen as

$$D = \{[z_0, z_1, z_2] \in \mathbb{CP}^2 | (z_0 + z_1 + z_2) z_0 z_1 z_2 = 0\}.$$

Equivalently,

$$\mathcal{P}^2 = \{(w_1, w_2) \in (\mathbb{C}^*)^2 | 1 + w_1 + w_2 \neq 0\}.$$

If the compact pair-of-pants is defined as $\bar{\mathcal{P}}^2 = \mathbb{CP}^2 \backslash \tilde{D}$, where $\tilde{D}$ denotes the union of small tubular open neighbourhoods of the 4 generic lines, then $\bar{\mathcal{P}}^2 \subset \mathcal{P}^2$, and the interior $\mathrm{int}(\bar{\mathcal{P}}^2)$ is diffeomorphic to $\mathcal{P}^2$. The boundary $\partial \bar{\mathcal{P}}^2$ consists of 6 copies of 2-torus T^2 and 4 copies of the total space F of the trivial S^1-bundle over $\mathcal{P}^1$. By composing with the fibration $\mathcal{P}^1 \to \mathrm{Y}$, $F \to \mathrm{Y}$ is a fibration with generic fibres T^2, and one singular fibre of shape $\ominus \times \bigcirc$ over the vertex of Y.