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Estimates on Weighted L^q-Norms of the Nonstationary 3D Navier-Stokes Flow in an Exterior Domain

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Abstract. This paper is devoted to estimates on weighted L^q-norms of the nonstationary 3D Navier-Stokes flow in an exterior domain. By multiplying the Navier-Stokes equation with a well selected vector field, an integral equation is derived, from which, we establish the weighted estimate $\||x|^\alpha u(t)\|_q \le C\Big(1+t^{\frac{\alpha}{2}+\varepsilon}\Big)t^{-\frac{3}{2}(1-\frac{1}{q})}$, $t>0$, where $0<\alpha\le 1$ and $\frac{3}{2}<q<\infty$, or $1<\alpha<2$ and $\frac{3}{3-\alpha}<q<\infty$, $0<\varepsilon<1$ is arbitrary, and $u_0\in L^3_\sigma(\Omega)$, $|x|^\alpha u_0\in L^1(\Omega)$ with $\|u_0\|_3$ sufficiently small. With the aid of the representation of the flow, we also prove that if in addition $u_0\in D_a^{1-1/b,b}$ for some $\frac{6}{5}\le a<\frac{3}{2}$ and $1<b<2$ with $\frac{3}{a}+\frac{2}{b}=4$, then the optimal estimate $\||x|^\alpha u(t)\|_q\le C\Big(1+t^{\frac{\alpha}{2}}\Big)t^{-\frac{3}{2}(1-\frac{1}{q})}$, $t>0$ holds, where $\alpha>0$ and $1<q<\infty$. Compared with the literature, here no extra restriction is laid on the range of the exponents α and q.

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Key Words: weighted estimates; navier-stokes flow; exterior domain; L^1-data.

1 Introduction

This paper is devoted to estimates on weighted L^q-norms of the nonstationary 3D Navier-Stokes flow in an exterior domain $\Omega\subseteq\mathbb{R}^3$, the exterior part of a bounded, single-connected and open set with smooth boundary $\partial\Omega$. Without loss of generality, the complement Ω^c is assumed contained in the unit ball $B_1(0)$ and $0\in\bar{\Omega}^c$. Recall that the Navier-Stokes flow can be described by the following initial-boundary problem

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$$\begin{cases}\partial_t u-\nu\Delta u+(u\cdot\nabla)u+\nabla p=0, & t>0, x\in\Omega,\\ \nabla\cdot u=0, & t>0, x\in\Omega,\\ u(t,x)=0, & t>0, x\in\partial\Omega,\\ u(t,x)\to 0 & \text{as } |x|\to\infty,\\ u(0,x)=u_0(x), & x\in\Omega.\end{cases} \tag{1.1}$$

Here $u_0(x)$ is a given solenoidal vector field, denoting the initial velocity, the vector-valued function $u(t,x)$ denotes the unknown velocity of the flow at the time $t>0$ and the point $x\in\Omega$, while the scalar function $p(x,t)$ stands for the unknown internal pressure of the fluid. For the sake of convenience, here the viscous coefficient is normalised to 1.

In the past decades, many authors paid attention to the existence and asymptotic behavior of the strong solution to (1.1). In general case $n\geq 2$, and Ω is the whole space $\mathbb{R}^n$, the half space $\mathbb{R}^n_+$, a bounded or an exterior domain, Kato [1], Kozono [2], Giga-Miyakama [3] and Iwashita [4] showed that if $u_0\in L^n(\Omega)$ and the norm $\|u_0\|_n\leq\eta_0$ for some small number $\eta_0>0$, then (1.1) has a unique and globally existing strong solution $u(x,t)$, called the Navier-Stokes flow. Iwashita [4], He-Miyakama [5], Han [6,7], Zhang-Zhu [8] etc investigated the time-decaying property for the L^q-norms of the flow in the exterior domain, under different initial conditions. Zhang-Zhang-Dong [9] gave regularity criteria for 3D Navier-Stokes equations.

We are much concerned about the estimates for the weighted L^q-norms of the flow. Because of the existence of the boundary of the fluid region, it is much involved to give suitable weighted estimates in exterior domains. By multiplying a well selected vector-valued function, He-Xin [10] removed the pressure term in the representation of the $3D$ flow, and derived that

$$\begin{aligned}&(1+|x|^2)^{\alpha/2}u\in L^\infty\big(0,\infty;L^q(\Omega)\big) && \text{for } \alpha=\tfrac{3}{7}-\tfrac{3}{q}, 7<q\leq\infty,\\ &(1+|x|^2)^{\beta/2}u\in L^\infty\big(0,\infty;L^q(\Omega)\big) && \text{for } \beta<1-\tfrac{3}{q}, 3<q\leq\infty,\end{aligned} \tag{1.2}$$

under the initial condition $u_0\in L^1(\Omega)\cap L^q_\sigma(\Omega)$ with $\|u_0\|_q$ sufficiently small, and $(1+|x|^2)^{\frac{\alpha}{2}}u_0\in L^q_\sigma(\Omega)$ or $(1+|x|^2)^{\frac{\beta}{2}}u_0\in L^q(\Omega)$.

With a modification of the method, some weighted estimates were given by Bae-Jin in [11,12] for $n=2,3$, and improved by Bae-Roh in [13], that is

$$\||x|^\alpha u(t)\|_q=O\Big(t^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})+\frac{\alpha}{2}+\varepsilon}\Big),\qquad t\to\infty,$$

where $0<\alpha<n$, $\frac{n}{n-\alpha}<q<\infty$, $1<r<n$, $u_0\in L^n_\sigma(\Omega)\cap L^r(\Omega)$, $|x|^\alpha u_0\in L^r(\Omega)$, and $\|u_0\|_n$ sufficiently small provided $n=3$. Based on the recognition that $|x|^\sigma u_0\in L^r(\Omega)$ implies $u_0\in L^s(\Omega)$ for some $1<s<r$, the undesired small number $\varepsilon>0$ was finally erased by Bae-Roh in [14], and optimal weighted estimates were derived, i.e.,

$$\||x|^\alpha u(t)\|_q=O\Big(t^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})+\frac{\alpha}{2}}\Big),\qquad t\to\infty,$$