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Asymptotic Behavior in a Quasilinear Two-Species Chemotaxis System with Nonlinear Sensitivity and Nonlinear Signal Secretion

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Abstract. In this paper, we study the asymptotic behavior of solutions to a quasilinear two-species chemotaxis system with nonlinear sensitivity and nonlinear signal production

$$\begin{cases} u_t=\nabla\cdot(D_1(u)\nabla u)-\nabla\cdot(S_1(u)\nabla v)+\mu_1 u(1-u-a_1w), & x\in\Omega,\ t>0,\\ \tau v_t=\Delta v-v+w^{\gamma_1}, & x\in\Omega,\ t>0,\\ w_t=\nabla\cdot(D_2(w)\nabla w)-\nabla\cdot(S_2(w)\nabla z)+\mu_2 w(1-w-a_2u), & x\in\Omega,\ t>0,\\ \tau z_t=\Delta z-z+u^{\gamma_2}, & x\in\Omega,\ t>0 \end{cases}$$

under homogeneous Neumann boundary conditions in a smooth bounded domain $\Omega\subset\mathbb{R}^n\,(n\geq 2)$ where the parameter $\mu_1,\mu_2,\gamma_1,\gamma_2$ are positive constants, $\tau\in\{0,1\}$. The diffusion coefficients $D_i,S_i\in C^2([0,\infty))$ satisfy $D_i(s)\geq a_0(s+1)^{-m_i}$, $0\leq S_i(s)\leq b_0 s(s+1)^{\beta_i-1}$, $s\geq 0$, $m_i,\beta_i\in\mathbb{R}$, $a_0,b_0>0$, $i=1,2$. Under the assumption of properly initial data regularity, we can find appropriate μ_i such that the globally bounded solution of this system satisfies the following relationship.
(I) If $a_1,a_2\in(0,1)$ and μ_1 and μ_2 are sufficiently large, then any globally bounded solution exponentially converges to $\left(\frac{1-a_1}{1-a_1a_2},\left(\frac{1-a_2}{1-a_1a_2}\right)^{\gamma_1},\frac{1-a_2}{1-a_1a_2},\left(\frac{1-a_1}{1-a_1a_2}\right)^{\gamma_2}\right)$ as $t\to\infty$;
(II) If $a_1>1>a_2>0$ and μ_2 is sufficiently large, then any globally bounded solution exponentially converges to $(0,1,1,0)$ as $t\to\infty$;
(III) If $a_1=1>a_2>0$ and μ_2 is sufficiently large, then any globally bounded solution algebraically converges to $(0,1,1,0)$ as $t\to\infty$.

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1 Introduction

Cell migration plays a significant role in a vast range of physiological and pathophysiological processes that take place within embryonic morphogenesis, wound healing, and tumor invasion. Chemotaxis, the tendency response of organisms to the stimulation of chemical substances in the external environment, is crucial for a variety of biological processes. It was the trailblazing work of chemotaxis model that Keller and Segel proposed in [1] to describe the accumulation of Dictyostelium discotylum, which was showed that

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v), & x \in \Omega,\ t > 0, \\ v_t = \Delta v - v + u, & x \in \Omega,\ t > 0. \end{cases} \tag{1.1}$$

The boundedness and blow-up of the solutions to the system (1.1) were studied by many scholars [2–6].

Subsequently, Keller and Segel presented the following chemotaxis model

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \chi(v) \nabla v) + \mu u(1-u), & x \in \Omega,\ t > 0, \\ v_t = \Delta v - v + f(u), & x \in \Omega,\ t > 0. \end{cases} \tag{1.2}$$

For $f(u) = u$ and $\mu = 0$, global solvability and blow-up of solutions were established in [7–9]. And, the global existence and the asymptotic behavior for (1.2) with logistic source were investigated [10–14]. When considering $f(u) = u(1+u)^{\beta-1}$ or a more general function, some researchers studied the global existence of solutions and the asymptotic behavior of solutions [15–18].

To better understand the model of two-species chemotaxis-competition system with two signals, it seems worthwhile to mention the following system,

$$\begin{cases} u_t = \Delta u - \chi_1 \nabla \cdot (u \nabla v) + \mu_1 u(1-u-a_1 w), & x \in \Omega,\ t > 0, \\ \tau v_t = \Delta v - v + w, & x \in \Omega,\ t > 0, \\ w_t = \Delta w - \chi_2 \nabla \cdot (w \nabla z) + \mu_2 w(1-w-a_2 u), & x \in \Omega,\ t > 0, \\ \tau z_t = \Delta z - z + u, & x \in \Omega,\ t > 0. \end{cases} \tag{1.3}$$

For $\tau = 0$ and $\mu_i = 0 (i = 1,2)$, in case $n \le 3$, Tao and Winkler [19] showed that this system has a unique globally bounded classical solution for any nonnegative initial data when $\chi_1 \in \{-1,1\}$ and $\chi_2 = -1$. When $\chi_1 = \chi_2 = 1$, if $n = 2$ and $m := \int_\Omega (u_0 + w_0) < C$ with some $C > 0$ or $n \ge 3$ and the initial data are sufficiently small, then globally bounded solutions were derived; if $n = 2$ and m is suitably large, or $n \ge 3$ and $m > 0$, then blow-up in finite time was obtained. For $\tau = 1$, in case $n = 2$, Black [20] obtained this model has global boundedness of solutions when the parameters in the system above are positive. Moreover, the author considered asymptotic stabilization of arbitrary globally bounded solution when $\frac{\mu_1}{\chi_1^2}$ and $\frac{\mu_2}{\chi_2^2}$ are sufficiently large. For more results about improvement of the model, the reader can refer to the literature [21–24].