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# Boundary Regularity for an Even Order Elliptic System in the Critical Dimension

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**Abstract.** In this short note, we consider the Dirichlet problem associated to an even order elliptic system with antisymmetric first order potential. Given any continuous boundary data, we show that weak solutions are continuous up to boundary.

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## 1 Introduction

In this paper, we consider the Dirichlet problem for the following even order elliptic system for $u\in W^{k,2}(\Omega,\mathbb{R}^m)$:

$$\Delta^k u=\sum_{l=0}^{k-1}\Delta^l\langle V_l,du\rangle+\sum_{l=0}^{k-2}\Delta^l\delta(w_l du)\quad \text{in } \Omega\subset\mathbb{R}^{2k} \tag{1.1}$$

with the following regularity assumptions on the coefficients:

$$\begin{cases} w_i\in W^{2i+2-k,2}(\Omega,\mathbb{R}^{m\times m}) \text{ for } i\in\{0,\dots,k-2\},\\ V_i\in W^{2i+1-k,2}\left(\Omega,\mathbb{R}^{m\times m}\otimes\wedge^1\mathbb{R}^{2k}\right) \text{ for } i\in\{1,\dots,k-1\}, \end{cases} \tag{1.2}$$

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and

$$V_0 = d\eta + F$$

with

$$\eta \in W^{2-k,2}(\Omega,\mathrm{so}(m)) \quad \text{and} \quad F \in W^{2-k,\frac{2k}{k+1},1}(\Omega,\mathbb{R}^{m\times m}\otimes\Lambda^1\mathbb{R}^{2k}). \tag{1.3}$$

This system was initially introduced by de Longueville and Gastel [1], aiming at a further extesion of the second order theory by Rivière [2] (corresponding to the case $k=1$) and the fourth order theory by Lamm-Rivière [3] (corresponding to the case $k=2$), addressing an open problem of Rivière. It includes the Euler-Lagrange equations of many interesting classes of geometric mappings such as the harmonic mappings, biharmonic mappings, polyharmonic mappings and so on; see [2–8].

A distinguished feature of this system is the *criticality*. To see it, we consider the simpler case $k=1$. Then system (1.1) reduces to the second order Rivière system

$$\Delta u = \Omega' \cdot \nabla u, \tag{1.4}$$

where $u \in W^{1,2}(\Omega,\mathbb{R}^m)$ and $\Omega' \in L^2(\Omega, so(m)\otimes\Lambda^1\mathbb{R}^2)$. The right hand side of (1.4) is merely in $L^1$ by Hölder's inequality and so standard $L^p$ regularity theory for elliptic equations fails to apply here. In the celebrated work [2], Rivière succeeded in rewriting (1.4) into an equivalent conservation law, from which the continuity of weak solutions follows. The techniques were further extended to fourth order system in [3] and finally to general even order systems in [1].

In this paper, we shall consider the Dirichlet boundary value problem for (1.1). Recall that we say that $u \in W^{k,2}(\Omega,\mathbb{R}^m)$ has Dirichlet boundary value $g \in C^{k-1}(\overline{\Omega},\mathbb{R}^m)$ if

$$\nabla^\alpha u = \nabla^\alpha g \quad \text{on } \partial\Omega$$

holds in the sense of traces for all $2k$-dimensional multi-indices $\alpha$ with $|\alpha| \le k-1$. Similarly, we say that $u$ has Navier boundary value $h_i \in C(\overline{\Omega},\mathbb{R}^m)$, $i=0,\cdots,k-1$, if for all $i \in \{0,\cdots,k-1\}$

$$\Delta^i u = h_i \quad \text{on } \partial\Omega.$$

Now, we can state our main theorem.

**Theorem 1.1.** *Fix $k \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^{2k}$ a bounded smooth domain. Suppose $u \in W^{k,2}(\Omega,\mathbb{R}^m)$ is a solution of* (1.1) *with* (1.2) *and* (1.3). *If either the Dirichlet boundary value $g \in C^{k-1}(\overline{\Omega},\mathbb{R}^m)$ or the Navier boundary value $h_i \in C(\overline{\Omega},\mathbb{R}^m)$ for $i=0,\cdots,k-1$, then $u \in C(\overline{\Omega},\mathbb{R}^m)$.*

Theorem 1.1 can be viewed as a natural extension of the corresponding boundary continuity results of Müller-Schikorra [9] for second order system, and Guo-Xiang [10] for fourth order system. As a special case of Theorem 1.1, we infer that every (extrinsic or intrinsic) polyharmonic mapping from the unit ball $B^{2k} \subset \mathbb{R}^{2k}$ into a closed manifold $N \hookrightarrow \mathbb{R}^m$ is continuous up the boundary, under the Dirichlet boundary value condition. This partially extends the corrosponding boundary continuity result of Lamm-Wang [11].