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Global and Non-Global Solutions for Pseudo-Parabolic Equation with Singular Potential

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Abstract. This paper considers initial boundary value to a pseudo-parabolic equation with singular potential $\frac{u_t}{|x|^s}-\Delta u_t-\Delta u=|u|^{p-2}u$ with $2<p<\frac{2N}{N-2}$, which was studied in [1] by Lian et al. They dealt with the global existence, asymptotic behavior with low initial level $J(u_0)\leq d$ and got the blow-up conditions of solutions with low and high initial level. In this paper, we give a new blow-up result which independent of the initial Nehari functional $I(u_0)$, and estimate the lower bound for blow-up time under some conditions. Finally, the precise exponential decay estimate is obtained for global solution with some conditions.

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Key Words: Pseudo-parabolic equation; singular potential; blow-up; bounds for blow up time; exponential decay.

1 Introduction

In this paper, we consider the initial boundary value problem for pseudo-parabolic equation with singular potential

$$\begin{cases} \frac{u_t}{|x|^s}-\Delta u_t-\Delta u=|u|^{p-2}u, & (x,t)\in\Omega\times(0,T), \\ u(x,0)=u_0(x), & x\in\Omega, \\ u(x,t)=0, & (x,t)\in\partial\Omega\times(0,T), \end{cases} \tag{1.1}$$

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where $2<p<\frac{2N}{N-2}$, $0\leq s\leq 2$, $\Omega\subset\mathbb{R}^N(N>2)$ is a bounded domain with smooth boundary $\partial\Omega$.

The pseudo-parabolic equation $cu_t=k\Delta u+ca\Delta u_t+f(x,t)$ widely apply on the field of physical, chemistry and biology, etc.. For example, to continuum thermodynamics with simple material, the coefficient c and k indicate the specific heat and conductivity respectively [2,3]. To seepage of homogeneous fluids through a fissured rock, the coefficient k indicate the size of the cracks in the rock. The decrease of the coefficient corresponds to the increase of the crack degree [4]. In addition, this model can be applied on population growth theory [5], semiconductor heat conduction theory [6] and long wave theory [7].

For the Cauchy problem

$$u_t-k\Delta u_t=\Delta u+u^p(x,t), \qquad (x,t)\ \in R^n\times(0,T),$$

Cao [8] obtained the critical Fujita exponent $p_c=1+\frac{2}{n}$ and Yang [9] gave the second critical exponent $a^*=\frac{2}{p-1}$. For nonlinear pseudo-parabolic, there are many papers have studied it. In this paper, we mainly use potential wells to study the properties of blow-up solution and global solution.

Sattinger [10] first proposed the potential well method in the initial boundary value problem

$$\begin{cases} u_t-\Delta u=f(u), & x\in\Omega,\ t>0,\\ u(x,0)=u_0(x), & x\in\Omega,\\ u(x,t)=0, & x\in\partial\Omega,\ t\geq 0,\end{cases} \tag{1.2}$$

to obtain the blow-up condition. From then on, this method became an important tool for studying the initial boundary value problem of nonlinear evolution equations, and got a lot of results [11]. Xu [12] and Liu [13] continued to give the condition of global and non-global solutions under the initial data $J(u_0)<d$ and $J(u_0)=d$ for problem (1.2).

To parabolic equation

$$\begin{cases} u_t-\Delta u=u^p, & (x,t)\in\Omega\times(0,T),\\ u(x,t)=0, & (x,t)\in\Omega\times(0,T),\\ u(x,0)=u_0(x)\geq 0, & x\in\Omega,\end{cases} \tag{1.3}$$

Payne [14] estimated the lower-upper bound of blow up time for nonglobal solution.

For a semilinear pseudo-parabolic equation

$$\begin{cases} u_t-\Delta u-\Delta u_t=u^p, & (x,t)\in\Omega\times(0,T),\\ u(x,t)=0, & (x,t)\in\partial\Omega\times(0,T),\\ u(x,0)=u_0(x)\geq 0, & x\in\Omega,\end{cases} \tag{1.4}$$