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Fine Behaviors of Eigenvalues and Eigenfunctions of the Lane-Emden Problem in Dimension Two

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Abstract. Recently, qualitative analysis of peaked solutions of Lane-Emden problem in dimension two has been widely considered. In particular, the Morse index of concentrated solutions with a single peak or multi peaks has been computed in [15, 16] separately. In this paper, we continue to consider the qualitative properties of the eigenvalues and eigenfunctions of the linearized Lane-Emden problem associated to peak solutions. Here we establish the fine behaviors of the first m eigenvalues and eigenfunctions of the linearized Lane-Emden problem in dimension two, and correspondingly the number of concentrated points of the first m eigenfunctions are given.

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1 Introduction and main results

Let us consider the following Lane-Emden problem

$$\begin{cases} -\Delta u = u^p, & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

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where $\Omega\subset\mathbb{R}^2$ and $1<p\to\infty$. The Lane-Emden equation models the mechanical structure of self-gravitating spheres, we refer to [1] for a physical introduction. From the mathematical point of view, problem (1.1) which looks extremely simple is an semilinear elliptic equation with a power focusing nonlinearity and it provides lots of interesting phenomena. Indeed, as we all know, the number of solutions to problem (1.1) strongly depends on the exponent p of the nonlinearity and on both the geometry and the topology of the domain Ω.

We can use standard variational methods to prove that problem (1.1) has at least one solution for any $p>1$, in any smooth bounded domain Ω. If p is close enough to 1, the solution is unique and nondegenerate in any domain Ω [2–4]. Moreover, if Ω is a ball [5] or, more in general, a symmetric and convex domain with respect to two orthogonal direction [2,6], the uniqueness and nondegeneracy hold true for any $p>1$. On the other hand, the multiplicity results for problem (1.1) have been proved in many cases, such as in some dumb-bell shaped domains at suitable values of p ([6]) or in annular domains (see for example [7–10]). We specially mention [11] where, in non simply connected domains, the existence of solutions of (1.1) concentrating at m points as $p\to+\infty$, for any $m\in\mathbb{N}$, is proven. After then, more attention stays qualitative analysis of peaked solutions of Lane-Emden equation. To state these results, we need to introduce some notations and recall the asymptotic characterization proved in [12] to state the fine behaviors of eigenvalues and eigenfunctions of the linearized Lane-Emden problem associated to concentrated solutions. We denote by $G(x,\cdot)$, the Green's function of $-\Delta$ in Ω, i.e. the solution to

$$\begin{cases}-\Delta G(x,\cdot)=\delta_x, & \text{in } \Omega,\\ G(x,\cdot)=0, & \text{on } \partial\Omega,\end{cases}$$

where δ_x is the Dirac function. We have the following decomposition formula of $G(x,y)$,

$$G(x,y)=\frac{1}{2\pi}\log|x-y|^{-1}+K(x,y) \quad \text{for} \quad (x,y)\in\Omega\times\Omega,$$

where $K(x,y)$ is the regular part of $G(x,y)$ and $\mathcal{R}(x)=K(x,x)$ is the Robin function. And then we recall some results concerning the asymptotic characterization of the positive solutions to (1.1) under the following energy uniform bound assumption:

$$\sup_p p\|\nabla u_p\|_2^2<\infty. \tag{1.2}$$

Theorem A ([12–14]). *Let u_p be a family of solutions to* (1.1) *satisfying* (1.2). *Let $R>0$ be such that $B_{2R}(\kappa_j)\subset\subset\Omega$, for $j=1,\cdots,m$ and $B_R(\kappa_i)\cap B_R(\kappa_j)=\emptyset$ if $i\neq j$. For each $\kappa_j\in S$, where $S=\{\kappa_1,\cdots,\kappa_m\}$,*

$$\lim_{p\to\infty} pu_p=8\pi\sqrt{e}\sum_{j=1}^{m}G(\cdot,\kappa_j) \quad \text{in} \quad C^{2,\alpha}_{loc}\left(\overline{\Omega}\setminus S\right), \tag{1.3}$$