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A Food Chain Model with a Protection Zone

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Abstract. In this paper, we are concerned about a food chain model with a protection zone for the prey species. Dynamical behavior, nonexistence and existence of positive steady states are obtained and there exist several critical values determined by the parameters and the protection zone for the growth rate of the prey species. The results reveal that the protection zone is effective for the survival of the prey species and beneficial for the coexistence of multiple species. Moreover, different properties of positive steady states from those of the two-species models are shown. The introduction of the prey or the top predator can be either favorable or unfavorable for the coexistence of multiple species.

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1 Introduction

The environment where the species live in is usually spatially heterogeneous, and ecosystems are profoundly affected by the spatial environment surrounding them [1,2]. However, the spatial heterogeneity brings new mathematical problems and it seems difficult to capture the influence of spatially heterogeneous environment. Much progress has been made on the study of spatial heterogeneity [3–8]. In particular, by introducing protection zone to reaction-diffusion systems, Du et. al. [9–11] revealed the effect of spatial heterogeneity on the predator-prey models and competition models. Biologically, human being sets up protection zone to save or protect certain species, where the certain species can

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enter and leave freely and the other species is blocked out. The introduction of a protection zone breaks the homogeneity of the environment of the species. For more study of protection zone, one can see [12–18] and references therein.

The food chain model is one of the important models describing interactions between multiple species, which has been widely studied [19–28]. It is known that the dynamics of food chain model is more complicated than that of two-species model, even for the ODE food chain model [29–33]. In this paper, we consider a food chain model with a protection zone for the prey species:

$$\begin{cases} u_t=\Delta u+u(\lambda-u-b(x)v), & x\in\Omega,\ t>0,\\ v_t=\Delta v+v(\mu-v+cu-dw), & x\in\Omega\setminus\overline{\Omega}_0,\ t>0,\\ w_t=\Delta w+w(-\gamma+\alpha v), & x\in\Omega\setminus\overline{\Omega}_0,\ t>0,\\ \partial_\nu u=0, & x\in\partial\Omega,\ t>0,\\ \partial_\nu v=\partial_\nu w=0, & x\in\partial\Omega\cup\partial\Omega_0,\ t>0, \end{cases} \tag{1.1}$$

where $u(x,t),v(x,t)$ and $w(x,t)$ represent the population densities of the prey, intermediate and top predators, respectively. The domain Ω is bounded and smooth in $\mathbb{R}^N(N\geq 1)$, Ω_0 is an open and connected subdomain of Ω with smooth boundary and $\overline{\Omega}_0\subset\Omega$. All the parameters are positive constants except μ and $b(x)$. The parameter μ is real constant and may take negative values. The function $b(x)$ is continuous and nonnegative with

$$b(x)\equiv 0,\quad x\in\Omega_0,\qquad b(x)>0,\quad x\in\overline{\Omega}\setminus\overline{\Omega}_0.$$

It is clear that Ω_0 is a protection zone for the prey u, while the intermediate and top predators are blocked out of Ω_0 and cannot enter Ω_0. For convenience of notations, we set $\Omega_1=\Omega\setminus\overline{\Omega}_0$.

In this paper, we are interested in the dynamical behavior and properties of positive steady states of (1.1). The principal eigenvalues play an important role in the analysis. Following [11], we let $\lambda_1^D(\phi,O)$ and $\lambda_1^N(\phi,O)$ be the principal eigenvalues of $-\Delta+\phi$ over the domain O with Dirichlet and Neumann boundary conditions respectively, where ϕ is a continuous function and O is a bounded domain with smooth boundary. If O is omitted, then we understand that $O=\Omega$; if ϕ is omitted, then we understand that $\phi=0$. It is known that for each fixed $\lambda\in(0,\lambda_1^D(\Omega_0))$, there exists a unique positive number $\mu_0=\mu_0(\lambda)$ such that

$$\lambda=\lambda_1^N(\mu_0 b(x)); \tag{1.2}$$

moreover, $\mu_0(\lambda)$ is strictly increasing with respect to $\lambda\in(0,\lambda_1^D(\Omega_0))$ and

$$\lim_{\lambda\to 0^+}\mu_0(\lambda)=0,\qquad \lim_{\lambda\to[\lambda_1^D(\Omega_0)]^-}\mu_0(\lambda)=+\infty.$$

The results reveal that there exist several critical values $0<\lambda_1^*<\lambda_2^*<\bar{\lambda}<\hat{\lambda}<\lambda_1^D(\Omega_0)$ determined by α,c,γ and the protection zone Ω_0 for the growth rate λ of the prey species.