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The Compactness of Extremals for a Singular Hardy-Trudinger-Moser Inequality

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Abstract. Motivated by a recent work of Wang-Yang [19] , we study the compactness of extremals $\{u_\beta\}$ for singular Hardy-Trudinger-Moser inequalities due to Hou [24] . In particular, by the method of blow-up analysis, we conclude that, up to a subsequence, u_β converges to an extremal in some sense as β tends to zero.

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1 Introduction

Let Ω be a smooth bounded domain in $\mathbb{R}^2$. The classical Trudinger-Moser inequality [1–5] writes

$$\sup_{u\in W_0^{1,2}(\Omega),||\nabla u||_2\leqslant 1}\int_\Omega e^{\gamma u^2}dx<\infty, \qquad \forall\gamma\leqslant 4\pi,$$

where $W_0^{1,2}(\Omega)$ denotes the standard Sobolev space and $||\cdot||_p$ denotes the usual L^p norm for any $p\geq 1$. There are lots of extensions of this inequality. On this topic, among others, we refer the readers to [6–12] and the references therein.

In particular, using a rearrangement argument, Adimurthi-Sandeep [7] proved that for any β, $0<\beta<1$, there holds

$$\sup_{u\in W_0^{1,2}(\Omega),||\nabla u||_2\leqslant 1}\int_\Omega \frac{e^{4\pi(1-\beta)u^2}}{|x|^{2\beta}}dx<\infty. \tag{1.1}$$

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Later Csató-Roy [13] concluded that the above supremum can be obtained by some extremal function $v_\beta \in W_0^{1,2}(\Omega)$ with $\|\nabla v_\beta\|_2 = 1$. A more general form was established by Yang-Zhu [14]. Precisely, let $\lambda_1(\Omega)$ be the first eigenvalue of the Laplacian operator with respect to the Dirichlet boundary condition, then for any $\alpha < \lambda_1(\Omega)$, there holds

$$\sup_{u \in W_0^{1,2}(\Omega), ||u||_{1,\alpha} \leqslant 1} \int_\Omega \frac{e^{4\pi(1-\beta)u^2}}{|x|^{2\beta}} dx < \infty, \tag{1.2}$$

where

$$\|u\|_{1,\alpha} = \left(\int_\Omega |\nabla u|^2 dx - \alpha \int_\Omega u^2 dx \right)^{\frac{1}{2}}$$

denotes a norm on $W_0^{1,2}(\Omega)$, which is equivalent to $\|\nabla u\|_2$. Also, they proved the existence of extremals for the supremum in (1.2). We mention that similar results still hold in the whole Euclidean space [15].

For any fixed $\beta \in (0,1)$, let u_β be an extremal function of the supremum in (1.2). One may ask whether (u_β) is pre-compact with respect to $\beta \in [0,1)$. Recently, Wang-Yang [19] demonstrated that, up to a subsequence, (u_β) converges to some u_0 in $C^1(\overline{\Omega})$ as $\beta \to 0$, where u_0 is an extremal function for the supremum in (1.2) with β=0. This partly answers the above-mentioned question.

In this paper, we concern the compactness of extremal functions for a Hardy-Trudinger-Moser inequality. Now we have fixed some notations. Let $\mathbb{B} \subset \mathbb{R}^2$ be the unit disc. Brezis-Marcus [16] improved the Hardy inequality

$$\int_{\mathbb{B}} |\nabla u|^2 dx \geqslant \int_{\mathbb{B}} \frac{u^2}{(1-|x|^2)^2} dx, \qquad \forall u \in W_0^{1,2}(\mathbb{B}),$$

into

$$\int_{\mathbb{B}} |\nabla u|^2 dx - \int_{\mathbb{B}} \frac{u^2}{(1-|x|^2)^2} dx \geqslant C \int_{\mathbb{B}} u^2 dx, \qquad \forall u \in W_0^{1,2}(\mathbb{B}), \tag{1.3}$$

for some constant $C > 0$. In view of (1.3),

$$||u||_{\mathscr{H}} = \left(\int_{\mathbb{B}} |\nabla u|^2 dx - \int_{\mathbb{B}} \frac{u^2}{(1-|x|^2)^2} dx \right)^{\frac{1}{2}} \tag{1.4}$$

defines a norm on $C_0^\infty(\mathbb{B})$. Let the function space $\mathscr{H}$ be a completion of $C_0^\infty(\mathbb{B})$ under the norm (1.4). Then, it is clear to see that $\mathscr{H}$ is a Hilbert space with the inner product

$$\langle u,v \rangle_{\mathscr{H}} = \int_{\mathbb{B}} \langle \nabla u, \nabla v \rangle dx - \int_{\mathbb{B}} \frac{uv}{(1-|x|^2)^2} dx.$$

In addition, the important fact

$$W_0^{1,2}(\mathbb{B}) \subset \mathscr{H} \subset \cap_{p \geqslant 1} L^p(\mathbb{B}) \tag{1.5}$$