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# Scattering for the Non-Radial Defocusing Nonlinear Inhomogeneous Hartree Equation

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**Abstract.** The purpose of this paper is to study scattering theory for the energy subcritical solutions to the non-radial defocusing inhomogeneous Hartree equation

$$i\partial_t u+\Delta u=(I_\alpha * |\cdot|^b|u|^p)|\cdot|^b|u|^{p-2}u.$$

Taking advantage of the decay factor in the nonlinearity instead of the embedding theorem, we establish the scattering criterion for the equation. Together with the Morawetz estimate, we obtain the scattering theory for the energy-subcritical case.

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**Key Words**: Inhomogeneous Hartree equation; scattering theory; Strichartz estimates; Morawetz estimate.

## 1 Introduction

In this paper, we consider the initial value problem for the inhomogeneous Hartree equation

$$\begin{cases} i\partial_t u+\Delta u=(I_\alpha * |\cdot|^b|u|^p)|\cdot|^b|u|^{p-2}u, & t\in\mathbb{R},\ x\in\mathbb{R}^N,\\ u(0,x)=u_0(x)\in H^1(R^N), \end{cases} \tag{1.1}$$

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where $u:\mathbb{R}\times\mathbb{R}^N\to\mathbb{C}$, $p\geq 2$ and $N\geq 3$. The factor of the inhomogeneous term $|\cdot|^b$ decays for some $b<0$. The Riesz potential is defined on $\mathbb{R}^N$ by

$$I_\alpha := \frac{\Gamma(\frac{N-\alpha}{2})}{\Gamma(\frac{\alpha}{2})\pi^{\frac{N}{2}}2^\alpha|\cdot|^{N-\alpha}} =: \frac{\mathcal{K}}{|\cdot|^{N-\alpha}}, \qquad 0<\alpha<N,$$

and

$$1+\frac{2+\alpha+2b}{N} < p < 1+\frac{2+\alpha+2b}{N-2}. \tag{1.2}$$

We assume that

$$\min\{2+\alpha+2b,\ 4+\alpha+2b-N\}>0. \tag{1.3}$$

The class of solutions to (1.1) is left invariant by the scaling

$$u_\lambda(t,x)=\lambda^{\frac{2+2b+\alpha}{2(p-1)}}u(\lambda^2 t,\lambda x), \tag{1.4}$$

which preserves the $\dot{H}^{s_c}(\mathbb{R}^N)$ norm with $s_c=\frac{N}{2}-\frac{2+2b+\alpha}{2(p-1)}$. Thus, we say that Eq. (1.1) is $\dot{H}^{s_c}(\mathbb{R}^N)$ critical. Moreover, Eq. (1.1) conserves the mass, defined by

$$M(u):=\int_{\mathbb{R}^N}|u|^2\mathrm{d}x,$$

and the energy, defined as the sum of the kinetic and potential energies:

$$E(u):=\int_{\mathbb{R}^N}\left[|\nabla u|^2+\frac{1}{p}(I_\alpha * |\cdot|^b|u|^p)|x|^b|u(x)|^p\right]\mathrm{d}x.$$

This class of equations models many physical phenomena, where the factor $|x|^b$ represents some inhomogeneity in the medium(see, e.g., [1]). Some particular cases of the above equation arise in the mean-field limit of large systems of non-relativistic atoms and molecules, the propagation of electromagnetic waves in plasmas, and so on. See [2–4] for more details. The local global well-posedness was obtained by Alharbi-Saanouni [5] via an adapted Gagliardo–Nirenberg type identity for the mass-energy intercritical. In [6], Saanouni-Xu studied the scattering theory following the argument of Dodson-Murphy [7] under the radial condition. Later, Xu [8] extended Saanouni's work to the non-radial case via a new approach established by Murphy [9].

The particular case $b=0$ of Eq. (1.1) is the nonlinear generalized Hartree equation. In [10], Feng–Yuan have studied the Cauchy problem for the generalized Hartree equation in the energy subcritical. Miao–Xu–Zhao [11,12] have studied the well-posedness issues of the Hartree equation which corresponds to the particular case $p=2$. Saanouni [13] proved the scattering theory for the radial case via a concentration-compactness method.

This work aims to study the scattering theory for the defocusing inhomogeneous Hartree equation. The well-posedness has been established in Alharbi-Saanouni [5]. Recently, Saanouni and Peng [14] proved the scattering under the radial setting. Now, we state the main result of this paper.