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Ground State Solutions for Kirchhoff Equations via Modified Nehari-Pankov Manifold

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Abstract. We investigate the Kirchhoff type elliptic problem

$$\left(a+b\int_{\mathbb{R}^N}[|\nabla u|^2+V(x)u^2]\mathrm{d}x\right)[-\Delta u+V(x)u]=f(x,u), \qquad x\in\mathbb{R}^N,$$

where both V and f are periodic in x, 0 belongs to a spectral gap of $-\Delta+V$. Under suitable assumptions on V and f with more general conditions, we prove the existence of ground state solutions and infinitely many geometrically distinct solutions.

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1 Introduction

Consider the following Schrödinger-Kirchhoff type nonlinear elliptic problem

$$\begin{cases}\left(a+b\int_{\mathbb{R}^N}[|\nabla u|^2+V(x)u^2]\mathrm{d}x\right)[-\Delta u+V(x)u]=f(x,u), \quad x\in\mathbb{R}^N,\\ u\in H^1(\mathbb{R}^N),\end{cases} \tag{1.1}$$

where $a>0$, $b>0$ are real constants, $N\geq 3$, $V:\mathbb{R}^N\to\mathbb{R}$ and $f:\mathbb{R}^N\times\mathbb{R}\to\mathbb{R}$.

Problem (1.1) is related to the stationary problem :

$$u_{tt}-\left(a+b\int_\Omega|\nabla u|^2\mathrm{d}x\right)\Delta u=f(x,u). \tag{1.2}$$

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Kirchhoff introduced the original form of (1.2) in 1883 to study the free vibration of the elastic strings. In (1.2), u denotes the displacement, $f(x,u)$ denotes the external force, and b denotes the initial tension while a is related to the intrinsic properties of the string, such as Young modulus. After that, Bernstein [1] first carried out the mathematical study on (1.2) in 1940. Moreover, after the works [2,3], many mathematicians have been discussing the solvability and asymptotic behavior of (1.2) for decades. There were also previous works on the nonlocal parabolic type problem involving the Dirichlet energy in [4,5]. Because of the lack of the compactness of the associated Sobolev embedding, a typical difficulty occurs in proving the Palais-Smale (PS) condition. Applying the pioneering argument by Brezis-Nirenberg [6] with the concentration compactness lemma by P.L Lions [7], the authors got existence results.

Recently, the Schröginger-Kirchhoff equation with periodic potentials and nonlinearities had found a great deal of interest in last years because not only it was important in applications but it provided a good model for developing mathematical methods (see [8-14] and the references therein). Liu et al. [9] considered a Kirchhoff type equation involving two potential by the Nehari manifold for the following Kirchhoff equation $-\left(a+b\int_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x\right)\Delta u+V(x)u=K(x)u^{2^*-1}$. Lin et al. [10] studied existence and concentration of ground state solutions for the following singularly perturbed Kirchhoff-type problem $-\left(\varepsilon^2 a+\varepsilon b\int_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x\right)\Delta u+V(x)u=f(u)$. In [11], the author had shown the effect of suitable singular potential $V(x)$ on the existence of multiple solutions of $-\Delta u=\lambda V(x)u+|u|^{2^*-2}u$ in bounded domain. In [13], the existence of (1.1) was obtained with the aid of the mountain pass theorem when $f(x,u)=u^q$ and $K(x)=1$. In [14], the author proved the multiple positive solutions.

Pankov [15] first introduced the Nehari-Pankov type manifold, which was a natural constraint and contained all nontrivial critical points. There were many results about the Nehari-Pankov type or non-Nehari manifold (see [16-29] and the references therein). For example, Tang et al. [23] studied the existence of ground state solutions of Nehari-Pankov to Schröginger systems, and infinitely many geometrically distinct solutions. In [28], the author proved the existence and asymptotical behavior of ground state solutions of Nehari-Pankov type to the above problem under some mild assumptions on V and f. In [29], assuming that the potential V was periodic and 0 lay in a spectral gap of $\sigma(-\Delta+V)$, least energy solution of the system was obtained for the super-quadratic case with a new technical condition,and the existence of ground state solutions of Nehari-Pankov type was established for the asymptotically quadratic case.

Motivated by the works [23] and [30], our goal is to study the existence of Nehari-Pankov type ground solutions of (1.1) with periodic potential, and obtain infinitely many geometrically distinct solutions for odd f under the more general conditions on f and V.

Precisely, $f(x,u)$ satisfies the following basic assumptions:
(V) $V\in\mathcal{C}(\mathbb{R}^N,\mathbb{R})$ is 1-periodic in each of $x_1,\cdots,x_N$ and

$$\sup[\sigma(-\Delta+V)\cap(-\infty,0)]<0<\bar{\Lambda}:=\inf[\sigma(-\Delta+V)\cap(0,\infty)]. \tag{1.3}$$