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A Weighted Trudinger-Moser Inequality and Its Extremal Functions in Dimension Two

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Abstract. Let Ω be a smooth bounded domian in $\mathbb{R}^2$, $H_0^1(\Omega)$ be the standard Sobolev space, and $\lambda_f(\Omega)$ be the first weighted eigenvalue of the Laplacian, namely,

$$\lambda_f(\Omega)=\inf_{u\in H_0^1(\Omega),\int_\Omega u^2\mathrm{d}x=1}\int_\Omega |\nabla u|^2 f\mathrm{d}x,$$

where f is a smooth positive function on Ω. In this paper, using blow-up analysis, we prove

$$\sup_{u\in H_0^1(\Omega),\int_\Omega |\nabla u|^2 f\mathrm{d}x\le 1}\int_\Omega e^{4\pi f u^2(1+\alpha\|u\|_2^2)}\mathrm{d}x<+\infty$$

for any $0\le\alpha<\lambda_f(\Omega)$. Furthermore, extremal functions for the above inequality exist when $\alpha>0$ is chosen sufficiently small.

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1 Introduction

Let Ω be a smooth bounded domain in $\mathbb{R}^2$, $H_0^1(\Omega)$ be the standard Sobolev space consisting of functions which vanish on $\partial\Omega$ and whose gradients are in $L^2(\Omega)$. Trudinger-Moser inequality plays an important role in conformal geometry and mathematical physics, readers are referred to surveys in [1,2]. The classical Trudinger-Moser inequality is attributed to Moser [3], Trudinger [4], Peetre [5], Pohozaev [6], Yudovich [7]. It says

$$\sup_{u\in H_0^1(\Omega),\int_\Omega |\nabla u|^2\mathrm{d}x=1}\int_\Omega e^{4\pi u^2}\mathrm{d}x<+\infty. \tag{1.1}$$

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Here 4π is the best constant in the sense that the above supremum is infinity if 4π is replaced by any larger number. Moreover, whether or not the supremum in (1.1) can be attained is another interesting question. Pioneer works in this regard were due to Carleson-Chang [1], Struwe [8], Flucher [9], Lin [10], Li [11, 12] and Yang [13]. The inequality (1.1) was improved by Adimurthi-Druet [14] to the following form: Let $\lambda(\Omega)>0$ be the first eigenvalue of the Laplace operator with respect to the Dirichlet boundary condition. Then for any $\alpha<\lambda(\Omega)$, there holds

$$\sup_{u\in H_0^1(\Omega),\int_\Omega|\nabla u|^2\mathrm{d}x=1}\int_\Omega e^{4\pi u^2(1+\alpha\|u\|_2^2)}\mathrm{d}x<+\infty. \tag{1.2}$$

The Riemann surface version of (1.2) was derived by Yang [15], who also obtained extremal functions for the above supremum in the case $\alpha<\alpha_0$ for some sufficiently small $\alpha_0>0$. Later, Lu-Yang [16] replaced $\|u\|_2$ with $\|u\|_p$ $(1<p<\infty)$ in (1.2) to get the same conclusion as in the case $p=2$. Also, similar result holds on Riemann surfaces [17]. The situation is quite different when the dimension $n\geq 3$. It was proved by Yang [18] that an analog of (1.2) still holds when Ω is a smooth bounded domain in $\mathbb{R}^n$. Extremal functions for such inequalities exist for all range $0\leq\alpha<\lambda_n(\Omega)$, where $\lambda_n(\Omega)>0$ is the first eigenvalue of the n-Laplace operator with respect to the Dirichlet boundary condition.

Let us come back to the 2-dimensional case. Compared with (1.2), a stronger version among others was shown by Tintarev [19], that is

$$\sup_{\int_\Omega|\nabla u|^2dx-\alpha\int_\Omega u^2\mathrm{d}x\leq 1}\int_\Omega e^{4\pi u^2}\mathrm{d}x<+\infty, \tag{1.3}$$

where $\alpha<\lambda(\mathbb{B}_R)$ with $|\mathbb{B}_R|=|\Omega|$. It was generalized by Yang [20] that the supremum in (1.3) is attained for all $\alpha<\lambda(\Omega)$. The case $n\geq 3$ was solved by Nguyen [21]. There are many other works in this topic, such as [22–25] and the references therein.

So far, we have not known whether or not extremal functions for (1.2) exists for general $\alpha<\lambda(\Omega)$. Recently, this problem was solved by Mancini-Thizy [26] via delicate energy estimates. They proved that there is no extremal function for $\alpha\in[\alpha_0,\lambda(\Omega))$, where α_0 is some number in $(0,\lambda(\Omega))$. This phenomenon also happens on Riemann surfaces [27, 28].

In this note, combining [29] and (1.2), we have an analog of [16]:

Theorem 1.1. *Let Ω be a smooth bounded domain in $\mathbb{R}^2$, f is a smooth positive function on Ω, and $\lambda_f(\Omega)$ is defined as*

$$\lambda_f(\Omega)=\inf_{u\in H_0^1(\Omega),\int_\Omega u^2\mathrm{d}x=1}\int_\Omega|\nabla u|^2 f\mathrm{d}x.$$

Then we have the following two conclusions:
(1) *For any $0\leq\alpha<\lambda_f(\Omega)$,*