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Adams-Onofri Inequality with Logarithmic Weight and the Associated Mean Field Bi-Harmonic Equation

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Abstract. This paper is devoted to establishing the Adams-Onofri inequality with logarithmic weight for the second order radial Sobolev space defined on the unit ball in $\mathbb{R}^4$. By using this inequality we obtain the existence of solutions for mean field biharmonic equation with logarithmic weight.

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Key Words: Adams-Onofri inequality; Logarithmic weight; mean field equation; existence.

1 Introduction

When $\Omega\subset\mathbb{R}^N$, $N\geq 2$ is a bounded domain, we have by the Sobolev embedding theorems that $W_0^{k,p}(\Omega)\hookrightarrow L^q(\Omega)$ $(1\leq q\leq \frac{Np}{N-kp},kp<N)$, and then $W_0^{k,\frac{N}{k}}(\Omega)\hookrightarrow L^q(\Omega)$ for any $1\leq q<\infty$. However, there are counterexamples show that $W_0^{k,\frac{N}{k}}(\Omega)\hookrightarrow L^\infty(\Omega)$ does not hold. In this case, it was proposed independently by Yudovic [1], Pohozaev [2] and Trudinger [3] that $W_0^{1,N}(\Omega)\subset L_{\varphi_N}(\Omega)$ where $\varphi_N(\Omega)$ is the Orlize space associated with the Young function $\varphi_N(t)=e^{|t|^{\frac{N}{N-1}}}$. This embedding was made more precise by Moser [4], which can be expressed by the famous Trudinger-Moser inequality:

$$\sup_{u\in W_0^{1,N}(\Omega),\|\nabla u\|^N_{L^N(\Omega)}\leq 1}\int_\Omega e^{\alpha u^{\frac{N}{N-1}}}\mathrm{d}x<\infty, \quad \text{if and only if} \quad \alpha\leq\alpha_N=N\omega_{N-1}^{\frac{1}{N-1}}, \tag{1.1}$$

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where ω_{N-1} is the area of the surface of the unit ball in $\mathbb{R}^N$.

There are many of literature on related results: extension of the Trudinger-Moser inequality to higher order Sobolev spaces or unbounded domains: see Adams [5], Li-Ruf [6], Adimurthi-Yang [7], Ruf-Sani [8], Lam-Lu [9], Chen-Lu-Zhu [10]; On compact Riemannian manifolds, see the work of Fontana [11], Li [12,13] and Yang [14]; Trudinger-Moser type inequalities on Heisenberg groups, see the work of Cohn-Lu [15], Lam-Lu [16], Yang [17] and Li-Lu-Zhu [18,19]; Trudinger-Moser inequalities involved in the Hardy term, see the work of Wang-Ye [20], Tintarev [21] and Ma-Wang-Yang [22]; Trudinger-Moser inequalities on the lines, see the work of Mancini-Martinazzi [23] and Chen-Wang-Zhu [24]; Trudinger-Moser inequalities under the Lorentz-Sobolev norms constraint, one can see the work [25–28].

In the famous work of Moser [4], the following inequality based on the Trudinger-Moser inequality on the two-dimensional sphere $\mathbb{S}^2$ was obtained: there exists some $c\geq 0$ such that:

$$\log\int_{\mathbb{S}^2} e^u \mathrm{d}\mu \leq \frac{1}{16\pi}\int_{\mathbb{S}^2}|\nabla u|^2\mathrm{d}\mu+\frac{1}{4\pi}\int_{\mathbb{S}^2} u\mathrm{d}\mu+c, \tag{1.2}$$

here $\mathrm{d}\mu$ denotes the volume element of two-dimensional sphere. Later, Onofri in [29] proved that the smallest of c above is 0. For this reason the inequality (1.2) is called Trudinger-Moser-Onofri inequality in some works. For more related results of the Trudinger-Moser-Onofri inequality, one can see the work of Dolbeault-Esteban [30].

In the literature, some weighted version of inequalities (1.1) and (1.2) were also studied. For instance, Calanchi-Ruf [31, 32] studied Trudinger-Moser type inequalities with logarithmic weights in dimension N, Later, By using main results of [32] and Young's inequality, the Trudinger-Moser-Onofri inequality with logarithmic weight was obtained in the work of Calanchi-Massa-Ruf [33], as described in the following:

Let $\mathbb{B}^2\subset\mathbb{R}^2$ denote the unit ball in $\mathbb{R}^2$, $w_\beta(x)=\left(\log\frac{e}{|x|}\right)^\beta$, $\beta\geq 0$, the authors in [33] considered the weighted sobolev space of radial functions

$$W^{1,2}_{0,rad}\left(\mathbb{B}^2,w_\beta(x)\right):=cl\left\{u\in C^\infty_{0,rad}\left(\mathbb{B}^2\right);\|u\|^2_{w_\beta}=\int_{\mathbb{B}^2}|\nabla u|^2 w_\beta(x)\mathrm{d}x<\infty\right\},$$

where cl stays for the closure of smooth radial functions with compact support in $\mathbb{B}^2$ under the norm $\|u\|_{w_\beta}$.

Theorem 1.1 ([33, Calanchi-Massa-Ruf]). *Let $\beta\in[0,1)$ and $\theta_\beta=\frac{2}{2-\beta}$. Then there exists a constant $C(\beta)$, such that*

$$\log\left(\frac{1}{|\mathbb{B}^2|}\int_{\mathbb{B}^2} e^{|u|^{\theta_\beta}}\mathrm{d}x\right)\leq\frac{1}{2\lambda}\|u\|^2_{w_\beta}+C(\beta), \tag{1.3}$$

for any $\lambda\leq\tilde{\lambda}^*_\beta=\pi(1-\beta)^\beta(2-\beta)^{2-\beta}2^{1-\beta}$, $u\in W^{1,2}_{0,rad}\left(\mathbb{B}^2,w_\beta(x)\right)$. This constant $\tilde{\lambda}^*_\beta$ is sharp in the sense that if $\lambda>\tilde{\lambda}^*_\beta$, then the above inequality can no longer hold.