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$C^{1,\alpha}$-Regularity for p-Harmonic Functions in SU(3)

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Abstract. This artical concerns the $C^{1,\alpha}_{\rm loc}$-regularity of weak solutions u to the degenerate subelliptic p-Laplacian equation

$$\triangle_{\mathcal{H},p}u(x)=\sum_{i=1}^{6}X_i^*(|\nabla_{\mathcal{H}}u|^{p-2}X_iu)=0,$$

where $\mathcal{H}$ is the orthogonal complement of a Cartan subalgebra in SU(3) with the orthonormal basis composed of the vector fields $X_1,\ldots,X_6$. When $1<p<2$, we prove that $\nabla_{\mathcal{H}}u\in C^{\alpha}_{\rm loc}$.

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1 Introduction

Denote by SU(3) the special unitary group of 3×3 complex matrices endowed with a horizontal vector field $\nabla_{\mathcal{H}}=\{X_1,\ldots,X_6\}$; see Section 2 for more geometries and properties of SU(3). Given a domain $\Omega\subset \mathrm{SU}(3)$, we consider the quasilinear subelliptic equation

$$\sum_{i=1}^{6}X_i^*(a_i(\nabla_{\mathcal{H}}u))=0 \quad \text{in } \Omega. \tag{1.1}$$

Here $\nabla_{\mathcal{H}}u=(X_1u,\ldots,X_6u)$ is the horizontal gradient of a function $u\in C^1(\Omega)$; X_i^* is the formal adjoint of X_i; the vector function $a:=(a_1,\ldots,a_6)\in C^2(\mathbb{R}^6,\mathbb{R}^6)$ satisfies the following

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growth and ellipticity conditions:

$$\sum_{i,j=1}^{6} \frac{\partial a_i(\xi)}{\partial \xi_j} \eta_i \eta_j \geq l_0 (\delta + |\xi|^2)^{\frac{p-2}{2}} |\eta|^2, \tag{1.2}$$

$$\sum_{i,j=1}^{6} \left| \frac{\partial a_i(\xi)}{\partial \xi_j} \right| \leq L (\delta + |\xi|^2)^{\frac{p-2}{2}}, \tag{1.3}$$

$$|a_i(\xi)| \leq L (\delta + |\xi|^2)^{\frac{p-2}{2}} |\xi| \tag{1.4}$$

for all $\xi, \eta \in \mathbb{R}^6$, where $0 \leq \delta \leq 1$, $1 < p < \infty$ and $0 < l_0 < L$. Note that conditions (1.2) and (1.3) are the same as conditions [1, (2.3) and (2.4)], but the condition (1.4) is stronger than the condition [1, (2.5)]

$$|a_i(\xi)| \leq L (\delta + |\xi|^2)^{\frac{p-1}{2}}. \tag{1.5}$$

We call a function $u \in W^{1,p}_{\mathcal{H},\mathrm{loc}}(\Omega)$ as a weak solution to (1.1) if

$$\sum_{i=1}^{6} \int_\Omega a_i(\nabla_{\mathcal{H}} u) X_i \varphi \mathrm{d}x = 0, \qquad \forall \varphi \in C_0^\infty(\Omega). \tag{1.6}$$

Here $W^{1,p}_{\mathcal{H},\mathrm{loc}}(\Omega)$ is the first order p-th integrable horizontal local Sobolev space, namely, all functions $u \in L^p_{\mathrm{loc}}(\Omega)$ with their distributional horizontal gradients $\nabla_{\mathcal{H}} u \in L^p_{\mathrm{loc}}(\Omega)$. Given the typical example $a(\xi) = (\delta + |\xi|^2)^{\frac{p-2}{2}} \xi$, Eq. (1.1) becomes the non-degenerate p-Laplacian equation

$$\sum_{i=1}^{6} X_i \big((\delta + |\nabla_{\mathcal{H}} u|^2)^{\frac{p-2}{2}} X_i u \big) = 0 \quad \text{if } \delta > 0, \tag{1.7}$$

and the p-Laplacian equation

$$\sum_{i=1}^{6} X_i \big(|\nabla_{\mathcal{H}} u|^{p-2} X_i u \big) = 0 \quad \text{if } \delta = 0. \tag{1.8}$$

Particularly, we call weak solutions to (1.8) as p-harmonic functions in $\Omega \subset \mathrm{SU}(3)$.

In the linear case $p = 2$, p-harmonic functions in SU(3) are usually called as harmonic functions and their C^∞-regularity was established by Hörmander [2]. In the quasilinear case $p \neq 2$, Domokos-Manfredi [1] obtained the local boundedness of horizontal gradient $\nabla_{\mathcal{H}} u$ of p-harmonic functions u in SU(3), that is, $\nabla_{\mathcal{H}} u \in L^\infty_{\mathrm{loc}}(\Omega)$. Moreover, when $2 < p < \infty$, they obtain the Hölder regularity of $\nabla_{\mathcal{H}} u$, that is, $\nabla_{\mathcal{H}} u \in C^{0,\alpha}(\Omega)$ for some $\alpha \in (0,1)$ independent of u. But when $1 < p < 2$, the Hölder regularity of $\nabla_{\mathcal{H}} u$ is unknown.

For the general quasi-linear equation (1.1) in SU(3), Domokos-Manfredi [1] also built up analogue regularity. To be precise, if a satisfies conditions (1.2), (1.3) and (1.5) for some