JOURNAL OF PARTIAL DIFFERENTIAL EQUATIONS
J. Part. Diff. Eq., Vol. **37**, No. 4, pp. 467-481

doi: 10.4208/jpde.v37.n4.6
December 2024

Global Existence and Stability for a Viscoelastic Wave Equation with Nonlinear Boundary Source Term

MELLAH Mohamed[1], HAKEM Ali[1,*] and LIU Gongwei[2]

[1] *Laboratory ACEDP, Djillali Liabes University, 22000 Sidi Bel Abbes, Algeria;*
[2] *College of Science, Henan University of Technology, Zhengzhou 450001, China.*

Received 25 September 2022; Accepted 24 August 2023

Abstract. This work considers the initial boundary value problem for a viscoelastic wave equation with a nonlinear boundary source term. Under suitable assumptions, we prove the existence of global weak solutions using the Galerkin approximation. Then, we give a decay rate estimate of the energy by making use of the perturbed energy method.

AMS Subject Classifications: 35B44, 35D30, 35L30, 35L82

Chinese Library Classifications: O175.27

Key Words: Viscoelastic equation; nonlinear boundary source; stabilization.

1 Introduction

In this work, we are concerned with the following initial boundary value problem

$$\begin{cases} u_{tt}-\Delta u-\Delta u_{tt}+\displaystyle\int_0^t g(t-s)\Delta u(s)\mathrm{d}s=0, & \text{in } \Omega\times\mathbb{R}^+, \\ u=0, & \text{on } \Gamma_0\times\mathbb{R}^+, \\ \dfrac{\partial u}{\partial \nu}+\dfrac{\partial u_{tt}}{\partial \nu}-\displaystyle\int_0^t g(t-s)\frac{\partial u}{\partial \nu}(s)\mathrm{d}s+|u|^{p-2}u=0, & \text{on } \Gamma_1\times\mathbb{R}^+, \\ u(x,0)=u_0(x), \quad u_t(x,0)=u_1(x), & \text{in } \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}^n$ with a smooth boundary Γ. Let $\{\Gamma_0, \Gamma_1\}$ be a partition of its boundary Γ such that $\Gamma=\Gamma_0\cup\Gamma_1$, $\overline{\Gamma_0}\cap\overline{\Gamma_1}=\emptyset$ and meas $(\Gamma_0)>0$. Here, $\frac{\partial}{\partial\nu}$ denotes

*Corresponding author. *Email addresses:* hakemali@yahoo.com (A. Hakem), m.mellah@univ-chlef.dz (M. Mellah), gongweiliu@haut.edu.cn (G. Liu)

the unit outer normal derivative to Γ. The term Δu_{tt} represents space-time dispersion. The function g represents the kernel of memory term, $p>2$.

Partial differential equations in viscoelastic materials have important physical background and important mathematical significance. The viscous effects are described and characterized by an integral term, and the integral term indicates a dissipative effect. For mathematical analysis on the motions of evolution equations with memory, we refer to [1, 2].

In the last few years, much effort in mathematics have been devoted to the study of qualitative properties for the viscoelastic equation

$$u_{tt}-\Delta u-\sigma\Delta u_{tt}+\int_0^t g(t-s)\Delta u(s)\mathrm{d}s=\mathcal{F}, \qquad x\in\Omega,\ t>0. \tag{1.2}$$

Eq. (1.2) and related quasi-linear problems with $|u_t|^\rho u_{tt}$ instead, $\rho>0$, have been extensively studied by many researches with possible external forces $\mathcal{F}$ like source $f(u)$ and damping $h(u_t)$. See for instance [3-14] and the references therein.

In 2008 Messaoudi [15,16] established a general decay of the energy solution to a viscoelastic equation corresponding to (1.2) with $\sigma=0$, by taking $\mathcal{F}=0$ and $\mathcal{F}=|u|^r u$, $r>0$. More precisely, he considered the following decay condition on the memory kernel

$$g'(t)\le -\xi(t)g(t), \qquad \forall t>0,$$

under proper conditions on $\xi(t)>0$, and proved general decay of energy such as

$$E(t)\le c_0 e^{-c_1\int_0^t \xi(s)\mathrm{d}s}, \qquad \forall t\ge 0,$$

for some $c_0,c_1>0$ depending on the initial data. Ever since several authors have used this condition to obtain arbitrary decay of energy for problems related to (1.2). See for instance the papers by Han and Wang [17,18], Liu [19], Liu and Sun [20], Park and Park [21]. Cavalcanti et al. [22] studied a problem of the form

$$\begin{cases} u_{tt}-\Delta u+\displaystyle\int_0^t g(t-s)\Delta u(s)\mathrm{d}s=0, & \text{in } \Omega\times\mathbb{R}^+,\\ u=0, & \text{on } \Gamma_0\times\mathbb{R}^+,\\ \dfrac{\partial u}{\partial \nu}-\displaystyle\int_0^t g(t-s)\frac{\partial u}{\partial \nu}(s)\mathrm{d}s+h(u_t)=0, & \text{on } \Gamma_1\times\mathbb{R}^+,\\ u(x,0)=u_0(x), \quad u_t(x,0)=u_1(x), & \text{in } \Omega, \end{cases} \tag{1.3}$$

for g,h specific functions and established uniform decay rate results under quite restrictive assumptions on both the damping function h and the kernel g. In fact, the function g had to behave exactly like e^{-mt} and the function h had a polynomial behavior near zero. For more general assumptions on g and h, Cavalcanti et al. [23] proved the uniform stability of (1.3) provided that $g(0)$ and $\|g\|_{L^1(0,\infty)}$ are sufficiently small. They also