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Large Time Behaviour for a Prey-Taxis System

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Abstract. The large time behaviour for a more general prey-axis system is considered. The asymptotically uniform boundedness of solutions in a suitable space is derived to ensure the dissipativity of the system. Based on the dissipativity of the system, the existence of a global attractor is obtained. The main technique used in this paper is the L^p-L^q estimation method.

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1 Introduction

This paper deals with the following parabolic system

$$\begin{cases} u_t=\Delta u-\nabla\cdot(uP(u,v)\nabla v)+G_1(u,v), & (x,t)\in\Omega\times(0,T),\\ v_t=\Delta v+G_2(u,v), & (x,t)\in\Omega\times(0,T),\\ \dfrac{\partial u}{\partial n}=\dfrac{\partial v}{\partial n}=0, & (x,t)\in\partial\Omega\times(0,T),\\ u(x,0)=u_0(x),v(x,0)=v_0(x), & x\in\Omega, \end{cases} \tag{1.1}$$

which was introduced by Karevia and Odell [1] to describe a direct motion of the predator u in response to a variation of the prey v, where $\Omega\subset\mathbb{R}^n(n\geq1)$ is a bounded domain with smooth boundary, the predator u and the prey v interact in terms of the functions $G_1(u,v)$ and $G_2(u,v)$, and the term $-\nabla\cdot(uP(u,v)\nabla v)$ reflects the prey-taxis mechanism.

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If $P(u,v)=0$, then (1.1) is reduced to a predator-prey model which has been widely studied [2–5]. Under the special case

$$G_1(u,v)=f(v)-u\Phi(v), \qquad G_2(u,v)=cu\Phi(v)-u(k+lu),$$

where $f(v)$ and $\Phi(v)$ satisfy some hypotheses, Wu, Wang and Shi [2] studied the global bifurcation and global stability of non-negative constant equilibrium solutions. With different nonlinearity, Du and Hsu [3] considered the positive steady state solutions and the global stability of non-negative constant equilibrium solutions. When $P(u,v)\neq 0$, various prey-taxis models have been studied in recent years to establish global existence, global boundedness and global stability of solutions [1,6,7]. With the special case $P(u,v)=\chi$ and

$$G_1(u,v)=\gamma uF(v)-uh(u), \qquad G_2(u,v)=-uF(v)+f(v),$$

the system (1.1) becomes

$$\begin{cases} u_t=\Delta u-\nabla\cdot(\chi u\nabla v)+\gamma uF(v)-uh(u), & (x,t)\in\Omega\times(0,T),\\ v_t=\Delta v-uF(v)+f(v), & (x,t)\in\Omega\times(0,T),\\ \dfrac{\partial u}{\partial n}=\dfrac{\partial v}{\partial n}=0, & (x,t)\in\partial\Omega\times(0,T),\\ u(x,0)=u_0(x),\ \ v(x,0)=v_0(x), & x\in\Omega. \end{cases} \tag{1.2}$$

The global classical solutions to (1.2) with global asymptotic behaviour of constant steady states have been obtained in [6] under the following hypotheses:

(H1) $F(v)\in C^2([0,\infty))$, $F(0)=0$, $F(v)>0$ in $(0,\infty)$ and $F'(v)>0$, $F''(v)\le 0$ on $[0,\infty)$;

(H2) the function $h:[0,\infty)\to(0,\infty)$ is continuously differentiable and there exist two constants $\theta>0$ and $\alpha\ge 0$, such that $h(u)\ge\theta$ and $h'(u)\ge\alpha$ for any $u\ge 0$;

(H3) the function $f:[0,\infty)\to\mathbb{R}$ is continuously differentiable satisfying $f(0)=0$, and there exist two constants $\mu,K>0$ such that $f(v)\le\mu v$ for any $v\ge 0$, $f(K)=0$ and $f(v)<0$ for all $v>K$. Moreover the ratio $\frac{f(v)}{F(v)}$ is continuous on $(0,\infty)$ and $\lim_{v\to 0}\frac{f(v)}{F(v)}$ exists.

It is mentioned that the conditions

$$\gamma F(K)\le\theta, \tag{1.3}$$

$$\gamma F(K)>\theta \quad \text{and} \quad \frac{D}{\chi^2}\ge D_c=\frac{u_*F^2(K)}{4\gamma F(v_*)F'(K)} \tag{1.4}$$

are crucial for the global asymptotic properties of constant steady states, which plays an important role in the Lyapunov functional procedure in [6], where (u_*,v_*) is the unique positive constant steady state of (1.2). On the other hand, as pointed out [6], it is still unclear whether pattern formation exists in more general cases, e.g., if (H1)-(H3) hold. Then a natural question is how to determine the large time behaviour of the system (1.2) in general, i.e., when system (1.2) can admit pattern formation.

The aim of this paper is to determine the asymptotic behaviour for the system (1.2) without conditions (1.3) and (1.4). Let

$$X:=\left\{\varphi\in W^{1,p}(\Omega):\varphi\geq 0 \text{ in } \Omega \text{ and } \frac{\partial\varphi}{\partial n}=0 \text{ on } \partial\Omega\right\},\qquad p\geq 1.$$

The main result is the following theorem.

Theorem 1.1. *Let $\Omega\subset\mathbb{R}^2$ and hypothesis* (H1)–(H3) *hold. Then* (1.2) *defines a dynamical system* (S_t,X) *associated with a global attractor $\mathcal{A}$ in X.*

Remark 1.1. Theorem 1.1 asserts the dissipative property and the existence of a compact invariant absorbing set of system (1.2) containing all of the steady states of system (1.2) under the assumptions (H1)–(H3), even if the conditions (1.3) and (1.4) required in [6] do not hold.

We will give some preliminary remarks in the next section, and then prove the dissipativity of the system (1.2) in Section 3 and the Theorem 1.1 in Section 4.

2 Preliminaries

We start with the global existence of classical solutions of (1.2) without proof. Refer to [6].

Lemma 2.1 ([6]). *Let $\Omega\subset\mathbb{R}^2$ be a smooth bounded domain and the hypotheses* (H1)-(H3) *hold. Assume $(u_0,v_0)\in[W^{1,p}(\Omega)]^2$ with $u_0,v_0\geq 0$ and $p>2$. Then problem* (1.2) *has a unique global classical solution $(u,v)\in C(\overline{\Omega}\times[0,\infty))\cap C^{2,1}(\overline{\Omega}\times(0,\infty))$ satisfying*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)}+\|v(\cdot,t)\|_{W^{1,\infty}(\Omega)}\leq C,$$

where $C>0$ is a constant independent of t, and in particular $0<v\leq K_0$ where

$$K_0:=\max\{\|v_0\|_{L^\infty},K\}.$$

Next, we introduce some well-known properties of the Neumann heat group. For $r\geq 1$, we denote $\partial\Omega$

$$A\varphi=-\Delta\varphi+\varphi \tag{2.1}$$

with

$$D(A)=\left\{\varphi\in W^{2,r}(\Omega):\frac{\partial\varphi}{\partial n}=0 \text{ on } \partial\Omega\right\},\qquad r\in(1,\infty). \tag{2.2}$$

Lemma 2.2 ([8]). *Let A be defined as in (2.1) and (2.2). Then the following estimates hold for all $\beta\geq 0$.*
(i) *If $1<p<\infty$, then*

$$\|A^{\beta}e^{-tA}w\|_{L^p(\Omega)}\leq ct^{-\beta}e^{-\nu_1 t}\|w\|_{L^p(\Omega)},\qquad t\in(0,T) \tag{2.3}$$

for any $w\in L^p(\Omega)$, and some $\nu_1>0$.
(ii) *If $1\leq p<q<\infty$, then*

$$\|A^{\beta}e^{-tA}w\|_{L^q(\Omega)}\leq ct^{-\beta-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}e^{-\mu t}\|w\|_{L^p(\Omega)},\qquad t\in(0,T) \tag{2.4}$$

holds for all $w\in L^q(\Omega)$, and some $\mu>0$.
(iii) *If $1<p<\infty$, then*

$$\|A^{\beta}e^{-tA}\nabla\cdot w\|_{L^p(\Omega)}\leq C(\varepsilon)t^{-\beta-\frac{1}{2}-\varepsilon}e^{-(1+\mu)t}\|w\|_{L^p(\Omega)},\qquad t\in(0,T) \tag{2.5}$$

is valid for all $w\in W^{1,q}(\Omega)$, and some $\mu>0$.

Lemma 2.3 ([6]). *Let Ω be a smooth bounded domain in $\mathbb{R}^n$. Let $1\leq p,q\leq\infty$ satisfy $(n-kq)p<nq$ for some $k>0$ and $r\in(0,p)$. Then for all $w\in W^{k,q}(\Omega)\cap L^r(\Omega)$, there exist two constants c_1 and c_2 depending only on Ω,q,k,r and n such that*

$$\|w\|_{L^p}\leq C\|D^k w\|_{L^q}^{a}\|w\|_{L^r}^{1-a}+C\|w\|_{L^r}$$

with $a\in(0,1)$ fulfilling

$$\frac{1}{p}=a\Big(\frac{1}{q}-\frac{k}{n}\Big)+(1-a)\frac{1}{r}.$$

Lemma 2.4 ([9]). *Let g,h,y be three positive locally integrable functions on $(t_0,+\infty)$ such that y' is locally integrable on $(t_0,+\infty)$, and which satisfies*

$$\frac{\mathrm{d}y}{\mathrm{d}t}\leq gy+h \qquad \text{for } t\geq t_0,$$
$$\int_t^{t+r}g(s)\mathrm{d}s\leq a_1,\quad \int_t^{t+r}h(s)\mathrm{d}s\leq a_2,\quad \int_t^{t+r}y(s)\mathrm{d}s\leq a_3,\qquad \text{for } t\geq t_0,$$

where r,a_1,a_2,a_3 are positive constants. Then

$$y(t+r)\leq\Big(\frac{a_3}{r}+a_2\Big)\exp(a_1),\qquad \forall t\geq t_0.$$

Lemma 2.5 ([10]). *Let α,β,γ be positive with $\beta+\gamma-1=\nu>0$, $\delta=\alpha+\gamma-1>0$, and*

$$u(t)\leq at^{\alpha-1}+b\int_0^t(t-s)^{\beta-1}s^{\gamma-1}u(s)\mathrm{d}s\qquad \text{for } t>0.$$

Then

$$u(t)\leq at^{\alpha-1}\sum_{m=0}^{\infty}C'_m(b\Gamma(\beta))^m t^{m\nu},$$

where $C'_0=1$, $\frac{C'_{m+1}}{C'_m}=\frac{\Gamma(m\nu+\delta)}{\Gamma(m\nu+\delta+\beta)}$.

3 Point dissipativity

In this section we establish the point dissipativity of the system (1.2), which is crucial for the main result. The key step is to prove the ultimately uniform boundedness of $\|u\|_{L^p(\Omega)}$ for any $p\geq 1$. This work is more difficult, we divide it into several Lemmas.

Denote by W a fixed, but arbitrarily bounded set in X, $\omega_i(t)$ continuous functions with property $\lim_{t\to\infty}\omega_i(t)=0$ $(i=1,\ldots,11)$, $C_i>0$ $(i=1,\ldots,5)$ independent of t, and $K_i>0$ $(i=1,\ldots,31)$ independent of the initial data and time t.

Using the comparison argument one can easily derive the ultimately uniform boundedness of $\|v\|_{L^\infty(\Omega)}$.

Lemma 3.1 ([6]). *Let* (u,v) *be a solution of* (1.2). *Then*

$$0\leq v(x,t)\leq K_1+\omega_1(t) \qquad \text{in } \Omega\times(0,T_{\max}). \tag{3.1}$$

The uniform boundedness of $\|u\|_{L^1(\Omega)}$ was also proved in [6], for the sake of uniformity, we prove it again in the ultimate uniform boundedness form.

Lemma 3.2. *Let* (u,v) *be a solution of* (1.2). *Then there exists* $C_1>0$ *such that*

$$\|u(\cdot,t)\|_{L^1(\Omega)}\leq K_2+\omega_2(t) \tag{3.2}$$

and

$$\int_t^{t+1}\|u(\cdot,s)\|^2_{L^2(\Omega)}\mathrm{d}s\leq K_3+\omega_3(t), \qquad \forall t\in(0,+\infty). \tag{3.3}$$

Proof. Integrating the equations for u and v in (1.2) gives

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u\mathrm{d}x=\gamma\int_\Omega uF(v)\mathrm{d}x-\int_\Omega uh(u)\mathrm{d}x,\\&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega v\mathrm{d}x=\int_\Omega f(v)\mathrm{d}x-\int_\Omega uF(v)\mathrm{d}x.\end{aligned} \tag{3.4}$$

Let $y(t)=\int_\Omega(u+\gamma v)\mathrm{d}x$. Observing (H1) and (H3), we then obtain

$$y'(t)+\theta y(t)\leq\gamma\theta\int_\Omega v\mathrm{d}x+\gamma\int_\Omega f(v)\mathrm{d}x\leq\gamma(\theta+\mu)(K_1+\omega_1(t)) \tag{3.5}$$

by using (3.1). Solving the differential inequality gives to

$$\begin{aligned}y(t)&\leq(\|u_0\|_{L^1(\Omega)}+\|v_0\|_{L^1(\Omega)})e^{-\theta t}+\gamma(\theta+\mu)\int_0^t e^{-\theta(t-s)}(K_1+\omega_1(s))\mathrm{d}s\\&\leq\gamma+\frac{\gamma\mu}{\theta}+(\|u_0\|_{L^1(\Omega)}+\|v_0\|_{L^1(\Omega)})e^{-\theta t}+\omega_3(t).\end{aligned} \tag{3.6}$$

This implies (3.2) with $K_2=\gamma+\frac{\gamma\mu}{\theta}$. From (H2) we deduce that $h(u)\geq\theta+\alpha u$, hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u\mathrm{d}x+\theta\int_\Omega u\mathrm{d}x+\frac{\alpha}{2}\int_\Omega u^2\mathrm{d}x\leq\frac{\gamma}{2\alpha}|\Omega|F^2(\|v(\cdot,t)\|_{L^\infty(\Omega)}). \tag{3.7}$$

Then (3.3) follows from (3.1), (3.2) and (3.7). □

Lemma 3.3. *Let* (u,v) *be a solution of* (1.2)*. Then there exist positive constants* K_4 *such that*

$$\limsup_{t\to\infty}\|\nabla v(\cdot,t)\|_{L^2(\Omega)}\leq K_4. \tag{3.8}$$

Proof. Denote by $\phi(u,v)=\gamma uF(v)-uh(u)$. Integrating the equation of u we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u\mathrm{d}x=\int_\Omega\phi(u,v).$$

Then integration over $[\tau,t]$ gives

$$\int_\tau^t\int_\Omega\phi(u,v)\mathrm{d}x\mathrm{d}s=\|u(\cdot,t)\|_{L^1(\Omega)}-\|u(\cdot,\tau)\|_{L^1(\Omega)}$$

with $\tau\geq 0$ to be determined later. This implies

$$\Big|\int_\tau^t\Big(\int_\Omega\phi(u,v)\mathrm{d}x\Big)\mathrm{d}s\Big|\leq\|u(\cdot,t)\|_{L^1(\Omega)}+\|u(\cdot,\tau)\|_{L^1(\Omega)}. \tag{3.9}$$

Since $\phi(u,v)\leq\gamma uF(v)-\alpha u^2$, we get

$$\int_\Omega u^2\mathrm{d}x\leq-\frac{1}{\alpha}\int_\Omega\phi(u,v)\mathrm{d}x+\frac{\gamma}{\alpha}\int_\Omega uF(v)\mathrm{d}x. \tag{3.10}$$

Multiplying the equation of v by $2v$ and $2\Delta v$ respectively gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega v^2\mathrm{d}x+\int_\Omega v^2\mathrm{d}x+2\int_\Omega|\nabla v|^2\mathrm{d}x\leq(2\mu+1)|\Omega|\|v\|^2_{L^\infty(\Omega)} \tag{3.11}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega|\nabla v|^2\mathrm{d}x\leq F^2(\|v\|_{L^\infty(\Omega)})\int_\Omega u^2\mathrm{d}x+|\Omega|f^2(\|v\|_{L^\infty(\Omega)}). \tag{3.12}$$

Therefore,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega(v^2+|\nabla v|^2)\mathrm{d}x+\int_\Omega(v^2+|\nabla v|^2)\mathrm{d}x\\ \leq&F^2(\|v\|_{L^\infty(\Omega)})\int_\Omega u^2\mathrm{d}x+|\Omega|f^2(\|v\|_{L^\infty(\Omega)})+(2\mu+1)|\Omega|\|v\|^2_{L^\infty(\Omega)}.\end{aligned} \tag{3.13}$$

Integrate (3.13) over $[\tau,t)$ to get

$$\|v(t)\|^2_{H^1(\Omega)}\leq\|v_\tau\|^2_{H^1(\Omega)}e^{-t}+\max_{s\geq\tau}F^2(\|v(s)\|_{L^\infty(\Omega)})\int_\tau^t e^{-(t-s)}\Big(\int_\Omega u^2\mathrm{d}x\Big)\mathrm{d}s$$

$$+\max_{s\geq\tau}\big(f^2(\|v(s)\|_{L^\infty(\Omega)})+(2\mu+1)\|v(s)\|_{L^\infty(\Omega)}\big)|\Omega|. \tag{3.14}$$

Inserting (3.9) and (3.10) into the (3.14), we get

$$\begin{aligned}\|v(t)\|^2_{H^1(\Omega)}\leq&\|v_\tau\|^2_{H^1(\Omega)}e^{-t}+\frac{1}{\alpha}\max_{s\geq\tau}F^2(\|v(s)\|_{L^\infty(\Omega)})(\|u(\cdot,t)\|_{L^1(\Omega)}+\|u(\cdot,\tau)\|_{L^1(\Omega)})\\&+\frac{\gamma}{\alpha}F^3(\|v(s)\|_{L^\infty(\Omega)})\int_\tau^t e^{-(t-s)}\Big(\int_\Omega u\mathrm{d}x\Big)\mathrm{d}s\\&+\max_{s\geq\tau}\big(f^2(\|v(s)\|_{L^\infty(\Omega)})+(2\mu+1)\|v(s)\|_{L^\infty(\Omega)}\big)|\Omega|.\end{aligned} \tag{3.15}$$

If we choose a $\tau>0$ such that $\max\{\omega_1(s),\omega_2(s)\}\leq 1$ when $s\geq\tau$ by Lemmas 3.1 and 3.2, then

$$\max\{\|v(\cdot,t)\|_{L^\infty(\Omega)},\ \|u(\cdot,t)\|_{L^1(\Omega)}\}\leq\max\{K_2,K_1\}+1,\qquad \text{for } \forall t\geq\tau. \tag{3.16}$$

Therefore (3.8) follows from (3.15) and (3.16). The proof is complete. □

Lemma 3.4. *Let (u,v) be a solution of (1.2). Then there are positive constants K_5 such that*

$$\limsup_{t\to\infty}\|u(\cdot,t)\|_{L^2(\Omega)}\leq K_5,\qquad \text{for } \forall t\in(0,\infty). \tag{3.17}$$

Proof. Multiplying the equation of u by $2u$ and integrating in Ω, we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^2\mathrm{d}x+2\int_\Omega|\nabla u|^2\mathrm{d}x\\ \leq&2\chi\int_\Omega u\nabla u\nabla v+2\gamma\int_\Omega F(v)u^2\mathrm{d}x-2\int_\Omega h(u)u^2\mathrm{d}x\\ \leq&\varepsilon\chi\int_\Omega|\nabla u|^2\mathrm{d}x+\chi\varepsilon^{-1}\int_\Omega u^2|\nabla v|^2\mathrm{d}x+2\gamma\int_\Omega F(v)u^2\mathrm{d}x-2\theta\int_\Omega u^2\mathrm{d}x-2\alpha\int_\Omega u^3\mathrm{d}x\end{aligned} \tag{3.18}$$

by using the Young's inequality. Apply the Young' inequality again to get

$$2\gamma F(v)u^2\leq\alpha u^3+\frac{2\gamma^3}{\alpha^2}F^3(v),$$

we then obtain by taking $\varepsilon=\frac{1}{\chi}$

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^2\mathrm{d}x+\int_\Omega|\nabla u|^2\mathrm{d}x\\ \leq&\chi^2\int_\Omega u^2|\nabla v|^2\mathrm{d}x+\frac{2\gamma^3}{\alpha^2}\int_\Omega F^3(v)\mathrm{d}x\\ \leq&\chi^2\int_\Omega u^2|\nabla v|^2\mathrm{d}x+\frac{2\gamma^3}{\alpha^2}|\Omega|F^3(\|v\|_{L^\infty(\Omega)}).\end{aligned} \tag{3.19}$$

Now we will estimate the item $\int_\Omega u^2|\nabla v|^2\mathrm{d}x$. Using the interpolation inequality with $n=2$

$$\|u\|_{L^4(\Omega)}\le\|u\|_{L^2(\Omega)}^{\frac12}\|u\|_{H^1(\Omega)}^{\frac12}=\|u\|_{L^2(\Omega)}^{\frac12}(\|\nabla u\|_{L^2(\Omega)}+\|u\|_{L^2(\Omega)})^{\frac12},$$

we derive

$$\begin{aligned}\int_\Omega u^2|\nabla v|^2\mathrm{d}x\le&\|u\|_{L^4(\Omega)}^2\|\nabla v\|_{L^4(\Omega)}^2\\ \le&\|u\|_{L^2(\Omega)}(\|\nabla u\|_{L^2(\Omega)}+\|u\|_{L^2(\Omega)})\|\nabla v\|_{L^4(\Omega)}^2\\ \le&\varepsilon\|\nabla u\|_{L^2(\Omega)}^2+C(\varepsilon)\|u\|_{L^2(\Omega)}^2(\|\nabla v\|_{L^4(\Omega)}^4+1).\end{aligned}\tag{3.20}$$

Inserting (3.20) into the (3.19) with $\varepsilon=\frac{1}{\chi^2}$, we easily get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^2\mathrm{d}x\le C(\varepsilon)\chi^2\|u\|_{L^2(\Omega)}^2(\|\nabla v\|_{L^4(\Omega)}^4+1)+\frac{2\gamma^3}{\alpha^2}|\Omega|F^3(\|v\|_{L^\infty(\Omega)}).\tag{3.21}$$

If we show

$$\limsup_{t\to\infty}\int_t^{t+1}\|\nabla v\|_{L^4(\Omega)}^4\mathrm{d}x\le K_6$$

with $K_6>0$ independent of u_0,v_0 and t, then (3.17) follows directly from Lemmas 2.4 and 3.2. Taking $\omega=\nabla v$, $q=2$, $r=4$ in the well-known interpolation inequality

$$\|\omega\|_{L^r(\Omega)}\le C_{q,r}\|\omega\|_{H^1(\Omega)}^{1-\frac{q}{r}}\|\omega\|_{L^q(\Omega)}^{\frac{q}{r}}$$

we get

$$\|\nabla v\|_{L^4(\Omega)}^4\le C\|\nabla v\|_{H^1(\Omega)}^2\|\nabla v\|_{L^2(\Omega)}^2\le\|\nabla v\|_{L^2(\Omega)}^2(\|\nabla v\|_{L^2(\Omega)}^2+\|\Delta v\|_{L^2(\Omega)}^2).\tag{3.22}$$

Now we are in the position to estimate $\int_t^{t+1}\|\Delta v\|_{L^2(\Omega)}^2\mathrm{d}s$. Multiplying the equation of v by v_t and integrating over Ω, we find that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega|\nabla v|^2\mathrm{d}x+\int_\Omega v_t^2\mathrm{d}x\le2\int_\Omega(v^2+u^2)\mathrm{d}x,$$

therefore,

$$\int_t^{t+1}\Big(\int_\Omega v_t^2\mathrm{d}x\Big)\mathrm{d}s\le\|\nabla v(\cdot,t)\|_{L^2(\Omega)}^2+2\int_t^{t+1}\Big(\int_\Omega(v^2+u^2)\mathrm{d}x\Big)\mathrm{d}s.\tag{3.23}$$

On the other hand, we can deduce from the equation for v that

$$\int_t^{t+1}\|\Delta v\|_{L^2(\Omega)}^2\mathrm{d}s\le C\Big(\int_t^{t+1}\|v_t\|_{L^2(\Omega)}^2\mathrm{d}s+\int_t^{t+1}\|u\|_{L^2(\Omega)}^2\mathrm{d}s+\int_t^{t+1}\|v\|_{L^2(\Omega)}^2\mathrm{d}s\Big),\tag{3.24}$$

which gives

$$\limsup_{t\to\infty}\int_t^{t+1}\|\Delta v\|_{L^2(\Omega)}^2 \mathrm{d}s \le K_7 \tag{3.25}$$

with some constant $K_7>0$, here we have used (3.3), (3.8), (3.23). Henceforth we have

$$\limsup_{t\to\infty}\int_t^{t+1}\|\nabla v\|_{L^4(\Omega)}^4 \mathrm{d}s \le K_8$$

by combining (3.23) with (3.22) and (3.25). The proof is complete. □

Now we present a differential inequality [11] which is crucial for the ultimately uniform boundedness of $\|u(\cdot,t)\|_p$ with $p\ge 1$. Consider the inequality

$$y'(t)\le \mathcal{F}(t,y), \quad y(0)=y_0, \qquad t\in(0,\infty) \tag{3.26}$$

with the property $y:\mathbb{R}^+\to\mathbb{R}$, suppose that:
(H4) There exists a continuous function $G(y,Y):\mathbb{R}^2\to\mathbb{R}$ such that for τ sufficiently large, if $t>\tau$ and $y(s)\le Y$ for every $s\in[\tau,t]$ then there exists $\tau'\ge\tau$ such that

$$\mathcal{F}(t,y)\le G(y(t),Y), \quad y(0)=y_0, \quad \text{if } t\ge\tau'\ge\tau. \tag{3.27}$$

(H5) The set $\{z:G(z,z)=0\}$ is not empty and $z_*=\sup\{z:G(z,z)=0\}<\infty$. Furthermore, $G(M,M)<0$ for all $M>z_*$.
(H6) For $y,Y\ge z_*$, $G(y,Y)$ is increasing in Y and decreasing in y.

Lemma 3.5 ([11]). *Assume* (3.26), (H4)–(H6). *If* $\limsup_{t\to\infty}y(t)<\infty$, *then*

$$\limsup_{t\to\infty} y(t)\le z_*.$$

Now we prove the ultimately uniform boundedness of $\|u(\cdot,t)\|_{L^p(\Omega)}$ in the following proposition by using an induction argument with the help of Lemma 3.5.

Proposition 3.1. *Let* (u,v) *be a solution of* (1.2). *Then there exist* K_9 *independent of t and initial values such that*

$$\limsup_{t\to\infty}\|u(\cdot,t)\|_{L^p(\Omega)}\le K_9 \qquad \text{for all } \ p\ge 1. \tag{3.28}$$

Proof. Obviously (3.28) holds when $p\in[1,2]$ by Lemmas 3.2 and 3.4. So, to prove (3.28), it suffices to show that the ultimately uniform boundedness of $\|u\|_{L^q(\Omega)}$ with $q\ge 2$ implies the the ultimately uniform boundedness of $\|u\|_{L^{2q}(\Omega)}$.

Denote $U:=u^q$. Multiply the equation of u in (1.1) by u^{2q-1} and integrate over Ω to get

$$\frac{1}{2q}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega U^2\mathrm{d}x+\frac{(2q-1)}{q^2}\int_\Omega|\nabla U|^2\mathrm{d}x$$

$$
\begin{aligned}
&=\chi(2q-1)\int_\Omega u^{2q-1}\nabla u\cdot\nabla v\mathrm{d}x+\gamma\int_\Omega U^2F(v)-\int_\Omega U^2h(u)\mathrm{d}x\\
&\le\frac{\chi\varepsilon(2q-1)}{2q^2}\int_\Omega|\nabla U|^2\mathrm{d}x+\frac{\chi(2q-1)}{2\varepsilon}\|\nabla v\|^2_{L^\infty(\Omega)}\int_\Omega U^2\mathrm{d}x\\
&\quad+\gamma F(\|v\|_{L^\infty(\Omega)})\int_\Omega U^2\mathrm{d}x-\int_\Omega U^2h(u)\mathrm{d}x.
\end{aligned}
$$

Choose $\varepsilon=\frac{1}{\chi}$, noticing $h(u)\ge\theta$ for any $u\ge 0$, we have

$$
\begin{aligned}
&\frac{1}{2q}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega U^2\mathrm{d}x+\theta\int_\Omega U^2\mathrm{d}x+\frac{(2q-1)}{2q^2}\int_\Omega|\nabla U|^2\mathrm{d}x\\
&\le\underbrace{\frac{\chi^2(2q-1)}{2}\|\nabla v\|^2_{L^\infty(\Omega)}\int_\Omega U^2\mathrm{d}x}_{I_1}+\underbrace{\gamma F(\|v\|_{L^\infty(\Omega)})\int_\Omega U^2\mathrm{d}x}_{I_2}.
\end{aligned}
$$

We then estimate I_1 and I_2 by using the inequality

$$
\int_\Omega U^2\mathrm{d}x\le\eta\left\{\int_\Omega|\nabla U|^2\mathrm{d}x+\|U\|^2_{L^1(\Omega)}\right\}+C\eta^{-1}\|U\|^2_{L^1(\Omega)}
$$

(see [11]) with $\eta=\frac{1}{4q^2\chi^2\|\nabla v\|^2_{L^\infty(\Omega)}}$ and $\eta=\frac{2q-1}{8\gamma q^2F(\|v\|_{L^\infty(\Omega)})}$ respectively to get

$$
\begin{aligned}
&I_1\le\frac{2q-1}{8q^2}\int_\Omega|\nabla U|^2\mathrm{d}x+C\frac{2q-1}{8q^2}\|U\|^2_{L^1(\Omega)}+2\chi^4q^2(2q-1)\|\nabla v\|^4_{L^\infty(\Omega)}\|U\|^2_{L^1(\Omega)},\\
&I_2\le\frac{2q-1}{8q^2}\int_\Omega|\nabla U|^2\mathrm{d}x+C\frac{2q-1}{8q^2}\|U\|^2_{L^1(\Omega)}+\frac{8\gamma^2q^2F^2(\|v\|_{L^\infty(\Omega)})}{(2q-1)}\|U\|^2_{L^1(\Omega)}.
\end{aligned}
$$

From this we derive that

$$
\begin{aligned}
&\frac{1}{2q}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega U^2\mathrm{d}x+\theta\int_\Omega U^2\mathrm{d}x+\frac{(2q-1)}{4q^2}\int_\Omega|\nabla U|^2\mathrm{d}x\\
&\le 2\chi^4q^2(2q-1)\|\nabla v\|^4_{L^\infty(\Omega)}\|U\|^2_{L^1(\Omega)}+C\left(\frac{2q-1}{4q^2}+\frac{8\gamma^2q^2F^2(\|v\|_{L^\infty(\Omega)})}{2q-1}\right)\|U\|^2_{L^1(\Omega)}.
\end{aligned}
$$

Denote

$$
\begin{aligned}
&\varpi_1(t)=C\left(\frac{2q-1}{2q}+\frac{16\gamma^2q^3F^2(\|v\|_{L^\infty(\Omega)})}{2q-1}\right)\|U\|^2_{L^1(\Omega)},\\
&\varpi_2(t)=4\chi^4q^3(2q-1)\|U\|^2_{L^1(\Omega)}.
\end{aligned}
$$

Let $\varphi=\int_\Omega U^2\mathrm{d}x$. Then we get

$$
\varphi'(t)\le -2q\theta\varphi(t)+\varpi_1(t)+\varpi_2(t)\|\nabla v\|^4_{L^\infty(\Omega)}. \tag{3.29}
$$

Now we deal with the estimate for $\|\nabla v\|^4_{L^\infty(\Omega)}$. Similar to (2.1) and (2.2), we define operator

$$A_1\varphi = -D\Delta\varphi + \varphi$$

with

$$D(A_1) = \left\{\varphi \in W^{2,q}(\Omega) : \frac{\partial\varphi}{\partial n} = 0 \text{ on } \partial\Omega\right\}, \qquad q \in (1,\infty).$$

Then the operator A_1 has fractional powers A_1^β, $\beta \geq 0$ whose domains of which have the embedding properties

$$D(A_1^\beta) \hookrightarrow C^\sigma(\overline{\Omega}) \text{ if } 2\beta - \frac{n}{q} > \sigma \geq 0.$$

Consider the abstract integrated version of the equation of v in $L^q(\Omega)$

$$v(t) = e^{-A_1 t}v_0 + \int_0^t e^{-A_1(t-s)}H(s)\mathrm{d}s, \qquad t > 0 \tag{3.30}$$

with $H(s) = -u(s)F(v(s)) + f(v(s)) + v(s)$.
If we choose $\sigma = 1$ and $n = 2$, then using Lemma 2.2 we have

$$\|\nabla v(t)\|_{L^\infty(\Omega)} \leq C_\beta t^{-\beta}e^{-\delta t}\|v_0\|_{L^q(\Omega)} + C_\beta\int_0^t (t-s)^{-\beta}e^{-\delta(t-s)}\|H(s)\|_{L^q(\Omega)}\mathrm{d}s. \tag{3.31}$$

By means of the interpolation inequality

$$\|u\|_{L^q(\Omega)} = \|u\|^{\frac{1}{q}}_{L^{\frac{r}{q}}(\Omega)} \leq \|U\|^{\frac{1}{q}-\vartheta}_{L^1(\Omega)}\|U\|^{2\vartheta}_{L^2(\Omega)}, \qquad \vartheta = \frac{\frac{1}{q}-\frac{1}{r}}{1-\frac{1}{2}}$$

with $r > q$. By choosing r close to q enough so that $4\vartheta < 1$, one can deduce that

$$\begin{aligned}\|H(s)\|_{L^q(\Omega)} \leq & F(\|v(s)\|_{L^\infty(\Omega)})\|u(s)\|_{L^q(\Omega)} + [f(\|v(s)\|_{L^\infty(\Omega)}) + \|v(s)\|_{L^\infty(\Omega)}] \\ \leq & F(\|v(s)\|_{L^\infty(\Omega)})\|U(s)\|^{\frac{1}{q}-\vartheta}_{L^1(\Omega)}\|U(s)\|^{2\vartheta}_{L^2(\Omega)} + [f(\|v(s)\|_{L^\infty(\Omega)}) + \|v(s)\|_{L^\infty(\Omega)}] \\ = & \varpi_3(s)\varphi^\vartheta(s) + \varpi_4(s).\end{aligned} \tag{3.32}$$

We conclude from (3.32) that

$$\begin{aligned}\|\nabla v(t)\|_{L^\infty(\Omega)} \leq & \omega_{10}(t) + C_\beta\int_0^t (t-s)^{-\beta}e^{-\delta(t-s)}[\varpi_3(s)\varphi^\vartheta(s) + \varpi_4(s)]\mathrm{d}s \\ \leq & \varpi_5(t) + C_\beta\int_0^t (t-s)^{-\beta}e^{-\delta(t-s)}\varpi_3(s)\varphi^\vartheta(s)\mathrm{d}s.\end{aligned} \tag{3.33}$$

For convenience, we write $\Phi(t) = \int_0^t (t-s)^{-\beta}e^{-\delta(t-s)}\varpi_3(s)\varphi^\vartheta(s)\mathrm{d}s$. Substituting (3.33) into (3.29), we get

$$\varphi'(t) \leq -2q\theta\varphi(t) + \varpi_6(t) + \varpi_2(t)\Phi^{4\vartheta}(t). \tag{3.34}$$

Now, we are in the position to check (H4)–(H6), and then (3.28) follows from Lemma 3.5.

For any fixed initial $(u_0,v_0)\in W$, there exists $\tau>0$ such that $\varpi_i(s)\le K_{10}$ $(i=2,3,6)$ if $s>\tau$. Suppose that $\Phi(s)\le\Psi$ for any $s\in[\tau,t]$, we can choose a $\tau_1>\tau$ to derive

$$\int_0^\tau (t-s)^{-\beta}e^{-\delta(t-s)}\varpi_3(s)\varphi^\vartheta(s)\mathrm{d}s\le 1$$

if $t>\tau_1$ due to the boundedness of $\|u\|_{L^\infty(\Omega)}$ dependent of (u_0,v_0) (Lemma 2.1). Therefore we have

$$\begin{aligned}\Phi(t)&=\int_0^\tau (t-s)^{-\beta}e^{-\delta(t-s)}\varpi_3(s)\varphi^\vartheta(s)\mathrm{d}s+\int_\tau^t (t-s)^{-\beta}e^{-\delta(t-s)}\varpi_3(s)\varphi^\vartheta(s)\mathrm{d}s\\&\le 1+\Psi^\vartheta\int_\tau^t (t-s)^{-\beta}e^{-\delta(t-s)}\mathrm{d}s\le 1+K_{10}K_{11}\Psi^\vartheta,\end{aligned}\tag{3.35}$$

where $K_{11}=\int_0^\infty (t-s)^{-\beta}e^{-\delta(t-s)}\mathrm{d}s<\infty$. Choose $r>\frac{2}{2\beta-1}>q$ close to q enough such that $\frac{1}{q}-\frac{1}{r}<\frac{1}{8}$, combing with (3.34), we arrive at $4\vartheta<1$ and

$$\begin{aligned}\varphi'(t)&\le -2q\theta\varphi(t)+K_{10}+K_{10}(1+K_{10}K_{11}\Psi^\vartheta)^4\\&\le -2q\theta\varphi(t)+2K_{10}+K_{12}\Psi^{4\vartheta}.\end{aligned}\tag{3.36}$$

Let

$$G(\varphi,\Psi)=-2q\theta\varphi(t)+9K_{10}+K_{12}\Psi^{4\vartheta},$$

then it is easy to see that $G(\varphi,\Psi)$ satisfies (H4), (H5) and (H6). Therefore we have

$$\limsup_{t\to\infty}\varphi(t)\le z_*\tag{3.37}$$

by Lemma 3.5, where z_* denotes the unique positive solution of the equation $-2q\theta\Psi+9K_{10}+K_{12}\Psi^{4\vartheta}=0$. Obviously z_* is independent of t and initials (u_0,v_0), we then have proved the ultimately uniform boundedness of $\|u(\cdot,t)\|_{L^p(\Omega)}$ for any $p\ge 1$. The proof is complete. □

4 Existence of a global attractor

In this section, using a semigroup argument inspired by [12, 13], we prove the global attractor to (1.2) based on the point dissipativity of the system (1.2).

Proof of Theorem 1.1. The global existence result guarantees that (1.2) admits a dynamical system. We start with the v-component. It follows from (3.30) that

$$\|v(t)\|_{\alpha'}\le\|A_1^{\alpha'-\gamma'}e^{-A_1t}v_0\|_{L^p(\Omega)}+C_{\alpha'}\int_0^t\|A_1^{\alpha'}e^{-A_1(t-s)}H(s)\|_{L^p(\Omega)}\mathrm{d}s$$

$$
\begin{aligned}
&\le C_{\alpha'} t^{\gamma'-\alpha'} e^{-\nu_1' t}\|v_0\|_{\gamma'} + C_{\alpha'}\int_0^t (t-s)^{-\alpha'} e^{-\nu_1'(t-s)}\|u(s)\|_{L^p(\Omega)} F\|v(s)\|_{L^\infty(\Omega)} \mathrm{d}s \\
&\qquad + C_{\alpha'}\int_0^t (t-s)^{\gamma'-\alpha'} e^{-\nu_1'(t-s)}\big(f(\|v(s)\|_{L^\infty(\Omega)}) + \|v(s)\|_{L^\infty(\Omega)}\big)\mathrm{d}s \\
&\le C_{\alpha'} t^{\gamma'-\alpha'} e^{-\nu_1' t}\|v_0\|_{\gamma'} + C_{\alpha'}\int_0^t (t-s)^{-\alpha'} e^{-\nu_1'(t-s)}(K_{13}+\omega_4(s))\mathrm{d}s \\
&\le C_{\alpha'} t^{\gamma'-\alpha'} e^{-\nu_1' t}\|v_0\|_{\gamma'} + K_{14} + \omega_5(t),
\end{aligned} \tag{4.1}
$$

where $0<\gamma'<1/2<\alpha'<1$. Therefore

$$
\limsup_{t\to\infty}\|v(t)\|_{\alpha'} \le K_{14}. \tag{4.2}
$$

Next, we prove the ultimate boundedness of $\|u(t)\|_\alpha$. Consider the parabolic equation

$$
\begin{cases}
\dfrac{\mathrm{d}\phi}{\mathrm{d}t} + A(t)\phi = \Phi(t), & x\in\Omega, t>0,\\
\dfrac{\partial\phi}{\partial n} = 0, & x\in\partial\Omega, t>0,\\
\phi(x,0) = \phi_0(x), & x\in\Omega,
\end{cases} \tag{4.3}
$$

where $A(t)\phi := -\Delta\phi + \phi$ with domain $D(A)$ as in (2.2), and Φ is Lipschitz continuous in ϕ. Rewrite the equation of u in (1.2) as

$$
u(t) = e^{-At}u_0 + \int_0^t e^{-A(t-s)}\Phi(s)\mathrm{d}s, \qquad t>0 \tag{4.4}
$$

with

$$
\Phi(s) = -\chi\nabla u(s)\cdot\nabla v(s) - \chi u(s)\Delta v(s) + u(s)[\gamma F(v(s)) - h(u(s)) + 1].
$$

From (4.4), we derive

$$
\begin{aligned}
\|u(t)\|_\alpha &\le \|A^\alpha e^{-At}u_0\|_{L^p(\Omega)} + \int_0^t \|A^\alpha e^{-(t-s)A}\Phi(s)\|_{L^p(\Omega)}\mathrm{d}s \\
&\le \underbrace{\|A^\alpha e^{-At}u_0\|_{L^p(\Omega)}}_{J_1} + \underbrace{C_\alpha\int_0^t (t-s)^{-\alpha}e^{-\nu_1(t-s)}\|\Phi(s)\|_{L^p(\Omega)}\mathrm{d}s}_{J_2}.
\end{aligned} \tag{4.5}
$$

We need to estimate J_1 and J_2 for any $t\in(0,T]$, where $T>0$ is to be fixed. Let $0<\alpha<1$. In the following, we denote by M_i the constant which dependents on $W\subset X$ and T. Then we have

$$
J_1 = \|A^\alpha e^{-At}u_0\|_{L^p(\Omega)} \le \|A^{\alpha-\gamma_1}e^{-At}A^{\gamma_1}u_0\|_{L^p(\Omega)} \le C_1 t^{\gamma_1-\alpha}e^{-\nu_1 t}\|u_0\|_{\gamma_1}
$$

with $0<\gamma_1\leq\alpha<1$.

Since we have the embedding [12]

$$W_B^{s,p}(\Omega)\hookrightarrow D(A^{\gamma_1})\qquad \text{for } \gamma_1<\frac{1}{2}\Big(s-\frac{n}{p}+\frac{n}{r}\Big)$$

with $W_B^{s,p}(\Omega)=\{\varphi\in W^{s,p}(\Omega):\frac{\partial\varphi}{\partial n}=0 \text{ on } \partial\Omega\}$ for $\frac{1}{p}<s\leq 1+\frac{1}{p}$, we immediately get the estimate

$$J_1=\|A^\alpha e^{-At}u_0\|_{L^p(\Omega)}\leq C_2 t^{\gamma_1-\alpha}e^{-\nu_1 t}\|u_0\|_{W^{1,p}(\Omega)}\leq M_1 t^{\gamma_1-\alpha}, \tag{4.6}$$

where $0<\gamma_1<\frac{1}{2}-\frac{1}{p}+\frac{1}{r}<\alpha<1$ and $r>p$.
Now, we estimate J_2. Note that,

$$\begin{aligned}\|\Phi(s)\|_{L^p(\Omega)}\leq&\chi\|\nabla u(s)\|_{L^p(\Omega)}\|\nabla v(s)\|_{L^\infty(\Omega)}+\chi\|u(s)\|_{L^q(\Omega)}\|\Delta v(s)\|_{L^{q'}(\Omega)}\\&+\|u(s)(\gamma F(v(s))-h(u(s))+1)\|_{L^p(\Omega)},\end{aligned}\tag{4.7}$$

where $\frac{1}{q}+\frac{1}{q'}=\frac{1}{p}$, $q,q'>p$. By Lemma 3.1 and Proposition 3.1 we find that

$$\|u(s)(\gamma F(v(s))-h(u(s))+1)\|_{L^p(\Omega)}\leq K_{15}+\omega_6(t). \tag{4.8}$$

Similar to (4.1), by (3.31) we assert that

$$\begin{aligned}\|H(\sigma)\|_{L^q(\Omega)}\leq&\|F(v(\sigma)\|_{L^\infty(\Omega)}\|u(\sigma)\|_{L^q(\Omega)}+\|f(v(\sigma))+v(\sigma)\|_{L^\infty(\Omega)}\\\leq&F(\|v(\sigma)\|_{L^\infty(\Omega)})\|u(\sigma)\|_{L^q(\Omega)}+(f(\|v(\sigma)\|_{L^\infty(\Omega)}+\|v(\sigma)\|_{L^\infty(\Omega)})\\\leq&K_{16}+\omega_7(t).\end{aligned}\tag{4.9}$$

In combination with (3.31) and the embedding $C^1(\Omega)\hookrightarrow D(A_r^{\alpha'})$ for $\alpha'>\frac{1}{2}+\frac{1}{r}$, we get

$$\begin{aligned}\|\nabla v(s)\|_{L^\infty(\Omega)}\leq&C_{\alpha'}s^{\gamma_1'-\alpha'}e^{-\nu_1' t}\|v_0\|_{\gamma_1'}+C_{\alpha'}\int_0^s(s-\sigma)^{-\alpha'}e^{-\nu_1'(s-\sigma)}\|H(\sigma)\|_{L^q(\Omega)}\mathrm{d}\sigma\\\leq&s^{\gamma_1'-\alpha'}\|v_0\|_{\gamma_1'}+C_{\alpha'}\int_0^s(s-\sigma)^{-\alpha'}e^{-\nu_1'(s-\sigma)}(K_{16}+\omega_7(\sigma))\mathrm{d}s\\\leq&(K_{17}+\omega_8(s))(1+s^{\gamma_1'-\alpha'}).\end{aligned}\tag{4.10}$$

Furthermore, the embedding $D(A^\beta)\hookrightarrow W^{1,r}(\Omega)$ gives

$$\|\nabla u(s)\|_{L^p(\Omega)}\leq C\|u(s)\|_\beta\leq C_{\beta,\varrho}\|u(s)\|_{L^p(\Omega)}^{1-\frac{\beta}{\alpha}}\|u(s)\|_\alpha^{\frac{\beta}{\alpha}} \tag{4.11}$$

with $1\leq p<r$, $\frac{1}{2}<\beta<\alpha$. We then derive

$$\|\nabla u(s)\|_{L^p(\Omega)}\|\nabla v(s)\|_{L^\infty(\Omega)}\leq C_{\beta,\varrho}\|u(s)\|_{L^p(\Omega)}^{1-\frac{\beta}{\alpha}}\|u(s)\|_\alpha^{\frac{\beta}{\alpha}}\|\nabla v(s)\|_{L^\infty(\Omega)}$$

$$\leq (K_{18}+\omega_9(s))(1+s^{\gamma_1'-\alpha'})\|u(s)\|_{\alpha}^{\frac{\beta}{\alpha}}. \tag{4.12}$$

Now we estimate $\|\Delta v(s)\|_{L^{q'}(\Omega)}$, to do so, using of the inequality [14]

$$\|\Delta v(s)\|_{L^{q'}(\Omega)} \leq C_{q'}\|A_1 v(s)\|_{L^{q'}(\Omega)}. \tag{4.13}$$

From (3.30) we deduce that

$$\begin{aligned}\|A_1 v(s)\|_{L^{q'}(\Omega)} &\leq C_{q'}\|A_1 e^{-A_1 s}v_0\|_{L^{q'}(\Omega)} + \int_0^s \|A_1 e^{-A_1(t-s)}H(s)\|_{L^{q'}(\Omega)}\mathrm{d}s \\ &\leq M_4 s^{\gamma_1'-1} + \int_0^t \|A_1^{1-\lambda}e^{-A_1(t-s)}A_1^{\lambda}H(s)\|_{L^{q'}(\Omega)}\mathrm{d}s \\ &\leq M_4 s^{\gamma_1'-1} + \int_0^s (s-\sigma)^{\lambda-1}e^{\nu_1'(s-\sigma)}\|A_1^{\lambda}H(s)\|_{L^{q'}(\Omega)}\mathrm{d}\sigma. \end{aligned} \tag{4.14}$$

Application of the continuous embedding $W^{s,q'}(\Omega) \hookrightarrow C^{\mu}(\Omega)$ with $0<2\lambda<s<\mu<1$ results in

$$\|A_1^{\lambda}H(\sigma)\|_{L^{q'}(\Omega)} \leq C_{\lambda}\|H(\sigma)\|_{W^{s,q'}(\Omega)} \leq C_{\lambda,\mu}\|H(\sigma)\|_{C^{\mu}(\Omega)}. \tag{4.15}$$

Furthermore, a simple calculation gives

$$\begin{aligned}\|H(\sigma)\|_{C^{\mu}(\Omega)} &\leq \|u(\sigma)F(v(\sigma))\|_{C^{\mu}(\Omega)} + \|f(v(\sigma))+v(\sigma)\|_{C^{\mu}(\Omega)} \\ &\leq \|u(\sigma)\|_{C^{\mu}(\Omega)}\|F(v(\sigma))\|_{C^{\mu}(\Omega)} + (1+\mu)\|v(\sigma)\|_{C^{\mu}(\Omega)} \\ &\leq (K_{19}+\omega_{10}(\sigma))(1+\|u(\sigma)\|_{C^{\mu}(\Omega)}), \end{aligned} \tag{4.16}$$

where $v \in C^{\mu}(\Omega)$ for all $\mu < 1-\frac{2}{r}$ by Proposition 5.1 [12]. Using the embedding $D(A^{\mu'}) \hookrightarrow C^{\mu}(\Omega)$ and the interpolation inequality $\|\phi\|_{\mu'} \leq C_{\mu',\gamma_2}\|\phi\|_{L^q(\Omega)}^{1-\vartheta'}\|\phi\|_{\gamma_2}^{\vartheta'}$ with $\frac{\mu}{2}+\frac{1}{r}<\mu'<\gamma_2<1$, we have

$$\|u(\sigma)\|_{C^{\mu}(\Omega)} \leq C_{\mu}\|u(\sigma)\|_{\mu'} \leq C_{\mu}\|u(\sigma)\|_{L^q(\Omega)}^{1-\vartheta'}\|u(\Omega)\|_{\gamma_2}^{\vartheta'} \tag{4.17}$$

with $\vartheta' = \frac{\mu'}{\gamma_2}$. This gives us

$$\|A^{\lambda}H(\sigma)\|_{L^{q'}(\Omega)} \leq (K_{20}+\omega_{11}(\sigma))(1+\|u(\sigma)\|_{\gamma_2}^{\vartheta'}) \tag{4.18}$$

and

$$\|A_1 v(s)\|_{L_{q'}(\Omega)} \leq M_4 s^{\gamma_1'-1}e^{-\nu_1' s} + M_5\int_0^s (s-\sigma)^{\lambda-1}e^{\nu_1'(s-\sigma)}(1+\|u(\sigma)\|_{\gamma_2}^{\vartheta'})\mathrm{d}\sigma. \tag{4.19}$$

For the estimate of J_2, one can easily deduce from Lemma 3.1 and Proposition 3.1 that

$$J_2 \leq \chi C_{\alpha}\int_0^t (t-s)^{-\alpha}e^{-\nu_1(t-s)}\|\nabla u(s)\|_{L^p(\Omega)}\|\nabla v(s)\|_{L^{\infty}(\Omega)}\mathrm{d}s$$

$$
\begin{aligned}
&+C_\alpha\int_0^t(t-s)^{-\alpha}e^{-\nu_1(t-s)}\|u(s)\|_{L^q(\Omega))}\|\Delta v(s)\|_{L^{q'}(\Omega)}\mathrm{d}s\\
&+C_\alpha\int_0^t(t-s)^{-\alpha}e^{-\nu_1(t-s)}\|u(s)(\gamma F(v(s))-h(u(s)+1]\|_{L^p(\Omega)}\mathrm{d}s\\
\le& K_{21}+\omega_{12}(t)+M_6\int_0^t(t-s)^{-\alpha}e^{-\nu_1(t-s)}(1+s^{\gamma_1'-\alpha'})\|u(s)\|_\alpha^{\frac{\beta}{\alpha}}\mathrm{d}s\\
&+M_7\int_0^t(t-s)^{-\alpha}e^{-\nu_1(t-s)}\mathrm{d}s\Big(s^{\gamma_1'-1}\\
&+\int_0^s(s-\sigma)^{\lambda-1}e^{-\nu_1'(s-\sigma)}(1+\|u(\sigma)\|_{\gamma_2}^{\vartheta'})\mathrm{d}\sigma\Big).
\end{aligned}\tag{4.20}
$$

Obviously, there exist positive constants $K_{19}>0$ and $\eta>0$ which satisfy

$$\|u(s)\|_\alpha^{\vartheta_1}\le K_{22}+\eta\|u(s)\|_\alpha.$$

Using the following equality

$$\int_\sigma^t(t-s)^{a-1}(s-\sigma)^{b-1}\mathrm{d}s=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}(t-\sigma)^{a+b-1}=C_{a,b}(t-\sigma)^{a+b-1}\tag{4.21}$$

we obtain

$$\int_0^t(t-s)^{-\alpha}e^{-\nu_1(t-s)}s^{\gamma_1'-1}\mathrm{d}s\le C_{\alpha,\gamma_1'}t^{\gamma_1'-\alpha}\tag{4.22}$$

and

$$
\begin{aligned}
&\int_0^t(t-s)^{-\alpha}e^{-\nu_1(t-s)}\mathrm{d}s\int_0^s(s-\sigma)^{\lambda-1}e^{-\nu_1'(s-\sigma)}(1+\|u(\sigma)\|_{\gamma_2}^{\vartheta'})\mathrm{d}\sigma\\
\le&\int_0^t e^{-\min\{\nu_1,\nu_1'\}(t-\sigma)}(1+\|u(\sigma)\|_{\gamma_2}^{\vartheta'})\mathrm{d}\sigma\int_\sigma^t(t-s)^{-\alpha}(s-\sigma)^{\lambda-1}\mathrm{d}s\\
\le&\int_0^t(t-\sigma)^{\lambda-\alpha}e^{-\min\{\nu_1,\nu_1'\}(t-\sigma)}\mathrm{d}\sigma+\int_0^t(t-\sigma)^{\lambda-\alpha}\|u(\sigma)\|_\alpha\mathrm{d}\sigma\\
\le& K_{23}+\int_0^t(t-\sigma)^{\lambda-\alpha}\|u(\sigma)\|_{\gamma_2}\mathrm{d}\sigma.
\end{aligned}\tag{4.23}
$$

Using the following inequality

$$a^pb^q\le\varepsilon(a+b)+C_\varepsilon$$

for positive constants p and q satisfying $p+q<1$, we have

$$(t-\sigma)^{\lambda-\alpha}=(t-\sigma)^\lambda(t-\sigma)^{-\alpha}\le C_{\lambda,\gamma}(t-\sigma)^{-\alpha}\sigma^{\gamma_1'-\alpha'}.\tag{4.24}$$

Therefore, we have

$$J_2\le M_8+M_9t^{\gamma_1'-\alpha}+M_{10}\int_0^t(t-s)^{-\alpha}s^{\gamma_1'-\alpha'}\|u(s)\|_{\gamma_2}\mathrm{d}s.\tag{4.25}$$

For any $t,s\in(0,T]$, with suitable choice of γ_1' and γ (e.g., we can choose $\gamma_1'>\gamma_1$), there exist positive constants κ_1 and M_{11} with respect to T such that

$$1+s^{\gamma_1'-\alpha'}\le\kappa_1 s^{\gamma_1'-\alpha'},\qquad M_8+M_1t^{\gamma_1-\alpha}+M_9t^{\gamma_1'-\alpha}\le M_{11}t^{\gamma_1-\alpha}.$$

Let $\alpha=\gamma_2$, we conclude from (4.6) and (4.25) that

$$\begin{aligned}\|u(t)\|_\alpha\le&\|A^\alpha e^{-At}u_0\|_{L^p(\Omega)}+\int_0^t\|A^\alpha e^{-(t-s)A}\Phi(s)\|_{L^p(\Omega)}\mathrm{d}s\\ \le&\underbrace{\|A^\alpha e^{-At}u_0\|_{L^p(\Omega)}}_{J_1}+\underbrace{C_\alpha\int_0^t(t-s)^{-\alpha}e^{-\nu_1(t-s)}\|\Phi(s)\|_{L^p(\Omega)}\mathrm{d}s}_{J_2}\\ \le&M_{11}t^{\gamma_1-\alpha}+M_{10}\int_0^t(t-s)^{-\alpha}s^{\gamma_1'-\alpha'}\|u(\sigma)\|_\alpha\mathrm{d}s\\ \le&M_{12}t^{\gamma_1-\alpha}\qquad\text{for any } t\in(0,T]\end{aligned}\tag{4.26}$$

by Lemma 2.5 if $\alpha+\alpha'<1+\gamma_1'$. Take notice that $\gamma'<\frac{1}{2}-\frac{1}{p}+\frac{1}{r}$, if we choose $\min\{\alpha,\alpha'\}\ge\frac{1}{2}+\frac{1}{r}$, then we get $r>\frac{2p}{p-2}$, which is sufficient for a suitable α in (4.26).

Choose $T\ge1$ so that $\omega_i(t)\le1$ $(i=1,\cdots,11)$ if $t\ge T$. Let T be the initial time, instead of (4.4), we consider the equation

$$u(t)=e^{-A(t-T)}u(T)+\int_T^t e^{-A(t-s)}\Phi(s)\mathrm{d}s.\tag{4.27}$$

In view of (2.3) and (4.26), we have

$$\begin{aligned}\|u(t)\|_\alpha\le&\|A^\alpha e^{-At}u(T)\|_{L^p(\Omega)}+\int_T^t\|A^\alpha e^{-A(t-s)}\Phi(s)\|_{L^p(\Omega)}\mathrm{d}s\\ \le&M_{13}e^{-\nu_1(t-T)}+C_{\alpha,\nu_1}\int_T^t(t-s)^{-\alpha}e^{-\nu_1(t-s)}\|\Phi(s)\|_{L^p(\Omega)}\mathrm{d}s.\end{aligned}\tag{4.28}$$

Similar to (4.7), we find

$$\begin{aligned}\|\Phi(s)\|_{L^p(\Omega)}\le&K_{15}+\omega_6(s)+(K_{18}+\omega_9(s))(1+s^{\gamma_1'-\alpha'})\|u(s)\|_\alpha^{\frac{\beta}{\alpha}}+\Big(M_4s^{\gamma_1'-1}e^{-\nu_1's}\\ &+\int_0^s(s-\sigma)^{\lambda-1}e^{-\nu_1'(s-\sigma)}(K_{20}+\omega_{11}(\sigma))(1+\|u(\sigma)\|_{\gamma_2}^{\vartheta'})\mathrm{d}\sigma\Big)\\ \le&K_{24}(1+\|u(s)\|_\alpha^{\frac{\beta}{\alpha}})+M_4e^{-\nu_1's}+\int_0^s(s-\sigma)^{\lambda-1}e^{-\nu_1'(s-\sigma)}\\ &\times(K_{20}+\omega_{11}(\sigma))(1+\|u(\sigma)\|_{\gamma_2}^{\vartheta'})\mathrm{d}\sigma,\qquad\forall s\ge T.\end{aligned}\tag{4.29}$$

Substituting (4.29) into the (4.28), we have the following estimates

$$\int_T^t(t-s)^{-\alpha}e^{-\nu_1(t-s)}(1+\|u(s)\|_\alpha^{\frac{\beta}{\alpha}})\mathrm{d}s\le K_{26}+\int_T^t(t-s)^{-\alpha}e^{-\nu_1(t-s)}\|u(s)\|_\alpha^{\frac{\beta}{\alpha}}\mathrm{d}s\tag{4.30}$$

and

$$\begin{aligned}\int_T^t (t-s)^{-\alpha} e^{-\nu_1(t-s)} e^{-\nu_1' s} \mathrm{d}s \leq & \int_T^t (t-s)^{-\alpha} e^{-\min\{\nu_1,\nu_1'\}(t-s)} \mathrm{d}s \\ \leq & e^{-\min\{\nu_1,\nu_1'\}\frac{t-T}{2}} \int_T^t (t-s)^{-\alpha} e^{-\min\{\nu_1,\nu_1'\}\frac{t-T}{2}} \mathrm{d}s \\ \leq & (\min\{\nu_1,\nu_1'\})^{\alpha-1} \Gamma(1-\alpha) e^{-\min\{\nu_1,\nu_1'\}\frac{t-T}{2}}. \end{aligned} \tag{4.31}$$

Exchanging the order of integral yields

$$\begin{aligned} & \int_T^t (t-s)^{-\alpha} e^{-\nu_1(t-s)} \mathrm{d}s \int_0^s (s-\sigma)^{\lambda-1} e^{-\nu_1'(s-\sigma)} (K_{20}+\omega_{11}(\sigma))(1+\|u(\sigma)\|_{\gamma_2}^{\vartheta'}) \mathrm{d}\sigma \\ \leq & \int_0^T \Big(\int_T^t (t-s)^{-\alpha} (s-\sigma)^{\lambda-1} \mathrm{d}s \Big) e^{-\min\{\nu_1,\nu_1'\}(t-\sigma)} (K_{20}+\omega_{11}(\sigma))(1+\|u(\sigma)\|_{\gamma_2}^{\vartheta'}) \mathrm{d}\sigma \\ & + \int_T^t \Big(\int_\sigma^t (t-s)^{-\alpha} (s-\sigma)^{\lambda-1} \mathrm{d}s \Big) e^{-\min\{\nu_1,\nu_1'\}(t-\sigma)} \\ & \times (K_{20}+\omega_{11}(\sigma))(1+\|u(\sigma)\|_{\gamma_2}^{\vartheta'}) \mathrm{d}\sigma. \end{aligned} \tag{4.32}$$

Because of (4.21) we have

$$\int_T^t (t-s)^{-\alpha} (s-\sigma)^{\lambda-1} \mathrm{d}s \leq \int_\sigma^t (t-s)^{-\alpha} (s-\sigma)^{\lambda-1} \mathrm{d}s \leq C_{1-\alpha,\lambda} (t-\sigma)^{\lambda-\alpha},$$

so

$$\int_T^t C_{1-\alpha,\lambda} (t-\sigma)^{\lambda-\alpha} e^{-\min\{\nu_1,\nu_1'\}(t-\sigma)} (K_{20}+\omega_{11}(\sigma)) \mathrm{d}\sigma \leq K_{27} \tag{4.33}$$

and

$$\begin{aligned} & \int_0^T C_{1-\alpha,\lambda} (t-\sigma)^{\lambda-\alpha} e^{-\min\{\nu_1,\nu_1'\}(t-\sigma)} (K_{20}+\omega_{11}(\sigma)) \mathrm{d}\sigma \\ \leq & M_{14} C_{1-\alpha,\lambda} e^{-\min\{\nu_1,\nu_1'\}\frac{t-T}{2}} \int_0^T (t-\sigma)^{\lambda-\alpha} e^{-\min\{\nu_1,\nu_1'\}\frac{t-\sigma}{2}} \mathrm{d}\sigma \\ \leq & M_{15} e^{-\min\{\nu_1,\nu_1'\}\frac{t-T}{2}} \end{aligned} \tag{4.34}$$

by using (4.26). Meanwhile

$$\begin{aligned} & \int_T^t \Big(\int_\sigma^t (t-s)^{-\alpha} (s-\sigma)^{\lambda-1} \mathrm{d}s \Big) e^{-\min\{\nu_1,\nu_1'\}(t-\sigma)} \|u(\sigma)\|_{\gamma_2}^{\vartheta'} \mathrm{d}\sigma \\ \leq & C_{1-\alpha,\lambda} \int_T^t (t-\sigma)^{\lambda-\alpha} e^{-\min\{\nu_1,\nu_1'\}(t-\sigma)} \|u(\sigma)\|_{\gamma_2}^{\vartheta'} \mathrm{d}\sigma \\ \leq & C_{1-\alpha,\lambda} \int_T^t (t-\sigma)^{-\alpha} e^{-\min\{\nu_1,\nu_1'\}\frac{t-\sigma}{2}} \|u(\sigma)\|_{\gamma_2}^{\vartheta'} \mathrm{d}\sigma, \end{aligned} \tag{4.35}$$

where we have used $x^\lambda e^{-\min\{\nu_1,\nu_1'\}x} \le e^{-\min\{\nu_1,\nu_1'\}\frac{x}{2}}$ for any $x\ge 0$ and $\lambda\in(0,1)$.

Combining these estimates, we have

$$\begin{aligned}\|u(t)\|_\alpha \le& M_{13}e^{-\nu_1(t-T)}+\int_T^t (t-s)^{-\alpha}e^{-\nu_1(t-s)}\|\Phi(s)\|_{L^p(\Omega)}\mathrm{d}s\\ \le& M_{16}e^{-\min\{\nu_1,\nu_1'\}\frac{(t-T)}{2}}+K_{28}+K_{29}\int_T^t (t-s)^{-\alpha}e^{-\min\{\nu_1,\nu_1'\}\frac{t-s}{2}}\|u(s)\|_\alpha^\vartheta \mathrm{d}s\end{aligned}$$

with $\beta\in(\frac{1}{2},\alpha)$ and $\vartheta=\frac{\beta}{\alpha}$, which implies

$$\limsup_{t\to\infty}\|u(t)\|_\alpha \le K_{30}\Big(1+\Big(\limsup_{t\to\infty}\|u(t)\|_\alpha\Big)^\vartheta\Big),$$

and thus $\limsup_{t\to\infty}\|u(t)\|_\alpha \le K_{31}$. Due to the compact embedding $D(A^{\alpha_1})\hookrightarrow W^{1,p}(\Omega)$ and $D(A_1^{\alpha_1})\hookrightarrow W^{1,p}(\Omega)$ for $\alpha_1=\min\{\alpha,\alpha'\}>\frac{1}{2}$, we assert the existence of the global attractor of (1.2) by [9, Theorem 1.1]. □

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