JOURNAL OF PARTIAL DIFFERENTIAL EQUATIONS
J. Part. Diff. Eq., Vol. **38**, No. 1, pp. 21-33

doi: 10.4208/jpde.v38.n1.2
March 2025

Multiple Positive Solutions for a Nonhomogeneous Schrödinger-Poisson System with Critical Exponent

ZHU Lijun[1,2] and LI Hongying[1,*]

[1] *School of Mathematics and Information, China West Normal University, Nanchong 637009, China;*
[2] *Nanchong Five-Start Middle School, Nanchong 637000, China.*

Received 17 May 2022; Accepted 23 May 2024

Abstract. In this paper, we consider the following nonhomogeneous Schrödinger-Poisson system

$$\begin{cases} -\Delta u+u+\eta\phi u=u^5+\lambda f(x), & x\in\mathbb{R}^3, \\ -\Delta\phi=u^2, & x\in\mathbb{R}^3, \end{cases}$$

where $\eta\neq 0$, $\lambda>0$ is a real parameter and $f\in L^{\frac{6}{5}}(\mathbb{R}^3)$ is a nonzero nonnegative function. By using the Mountain Pass theorem and variational method, for λ small, we show that the system with $\eta>0$ has at least two positive solutions, the system with $\eta<0$ has at least one positive solution. Our result generalizes and improves some recent results in the literature.

AMS Subject Classifications: 35B33, 35J20, 35J60

Chinese Library Classifications: O177.91

Key Words: Schrödinger-Poisson system; critical exponent; variational method; positive solutions.

1 Introduction

In this paper, we study the existence and multiplicity of positive solutions for the following nonhomogeneous Schrödinger-Poisson system with critical exponent

$$\begin{cases} -\Delta u+u+\eta\phi u=u^5+\lambda f(x), & x\in\mathbb{R}^3, \\ -\Delta\phi=u^2, & x\in\mathbb{R}^3, \end{cases} \tag{1.1}$$

*Corresponding author. *Email addresses:* lihongyingnch@163.com (H. Y. Li)

where $\eta \neq 0, \lambda > 0$ is a real parameter and $f \in L^{\frac{6}{5}}(\mathbb{R}^3)$ is a nonzero nonnegative function.

It is well known that the Schrödinger-Poisson system stems from quantum mechanics models and semiconductor theory (see [1–3]) and it has been studied extensively. From a physical standpoint, Schrödinger-Poisson systems describe systems of identical charged particles interacting each other if magnetic effects could be ignored and their solutions are standing waves. For more details about the mathematical and physical background of Schrödinger-Poisson system, we can refer to the papers [4-7] and the references therein.

Zhao and Zhao [8] studied the following Schrödinger-Poisson system with critical exponent for the first time

$$\begin{cases} -\Delta u + u + \phi u = K(x)|u|^4 u + \mu Q(x)|u|^{q-2}u, & x \in \mathbb{R}^3, \\ -\Delta \phi = u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.2}$$

where $2<q<6, \mu>0$ and $K, Q \in C(\mathbb{R}^3, \mathbb{R})$ satisfies some certain conditions. When $2<q<4$ and K, Q are radial functions with some certain conditions, they obtained system (1.2) has at least a positive radial solution for $\mu>0$ large enough; when $q=4$, they obtained system (1.2) possesses a positive solution for $\mu>0$ large enough; while $4<q<6$ they obtained system (1.2) has at least a positive solution for all $\mu>0$. Recently, Lei-Liu-Chu-Suo [9] considered the following Schrödinger-Poisson system

$$\begin{cases} -\Delta u + u + \eta \phi u = u^5 + \lambda f(x) u^{q-1}, & x \in \mathbb{R}^3, \\ -\Delta \phi = u^2, & x \in \mathbb{R}^3, \end{cases}$$

where $1<q<2, \eta \in \mathbb{R}\backslash\{0\}, \lambda>0$ is a real parameter and $f \in L^{\frac{6}{6-q}}(\mathbb{R}^3)$ is a nonzero nonnegative function. Using the variational methods, they obtained that there exists a positive constant λ_* such that for all $\lambda \in (0, \lambda_*)$, the system has at least two positive solutions. A natural question is whether there exist solutions for the critical Schrödinger-Poisson system with nonhomogeneous term (that is, the case of $q=1$ in system (1.2)). Ye [10] studied the following a class of nonhomogeneous Schrödinger-Poisson system

$$\begin{cases} -\Delta u + u + \lambda \phi u = f(u) + h(x), & x \in \mathbb{R}^3, \\ -\Delta \phi = u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.3}$$

where $\lambda>0$ is a parameter and $0 \leq h(x) = h(|x|) \in L^2(\mathbb{R}^3), f$ satisfies the following hypotheses:

(f_1) $f \in C(\mathbb{R}, \mathbb{R}^+), f(0)=0, f(t) \equiv 0$ for $t<0$ and there exist $a>0$ and $q \in (2,6)$ such that

$$f(t) \leq a(1+|t|^{q-1}), \qquad \forall t \in \mathbb{R}.$$

(f_2) $\lim_{t \to 0} \frac{f(t)}{t} = 0$.

(f_3) $\lim_{t \to \infty} \frac{f(t)}{t} = +\infty$.

She proved that system (1.3) has at least two positive solutions with the aid of Ekeland's