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Multiple Positive Solutions for a Nonhomogeneous Schrödinger-Poisson System with Critical Exponent

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Abstract. In this paper, we consider the following nonhomogeneous Schrödinger-Poisson system

$$\begin{cases} -\Delta u+u+\eta\phi u=u^5+\lambda f(x), & x\in\mathbb{R}^3,\\ -\Delta\phi=u^2, & x\in\mathbb{R}^3, \end{cases}$$

where $\eta\neq 0$, $\lambda>0$ is a real parameter and $f\in L^{\frac{6}{5}}(\mathbb{R}^3)$ is a nonzero nonnegative function. By using the Mountain Pass theorem and variational method, for λ small, we show that the system with $\eta>0$ has at least two positive solutions, the system with $\eta<0$ has at least one positive solution. Our result generalizes and improves some recent results in the literature.

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1 Introduction

In this paper, we study the existence and multiplicity of positive solutions for the following nonhomogeneous Schrödinger-Poisson system with critical exponent

$$\begin{cases} -\Delta u+u+\eta\phi u=u^5+\lambda f(x), & x\in\mathbb{R}^3,\\ -\Delta\phi=u^2, & x\in\mathbb{R}^3, \end{cases} \tag{1.1}$$

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where $\eta \neq 0, \lambda > 0$ is a real parameter and $f \in L^{\frac{6}{5}}(\mathbb{R}^3)$ is a nonzero nonnegative function.

It is well known that the Schrödinger-Poisson system stems from quantum mechanics models and semiconductor theory (see [1–3]) and it has been studied extensively. From a physical standpoint, Schrödinger-Poisson systems describe systems of identical charged particles interacting each other if magnetic effects could be ignored and their solutions are standing waves. For more details about the mathematical and physical background of Schrödinger-Poisson system, we can refer to the papers [4-7] and the references therein.

Zhao and Zhao [8] studied the following Schrödinger-Poisson system with critical exponent for the first time

$$\begin{cases} -\Delta u + u + \phi u = K(x)|u|^4 u + \mu Q(x)|u|^{q-2}u, & x \in \mathbb{R}^3, \\ -\Delta \phi = u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.2}$$

where $2 < q < 6, \mu > 0$ and $K, Q \in C(\mathbb{R}^3, \mathbb{R})$ satisfies some certain conditions. When $2 < q < 4$ and K, Q are radial functions with some certain conditions, they obtained system (1.2) has at least a positive radial solution for $\mu > 0$ large enough; when $q = 4$, they obtained system (1.2) possesses a positive solution for $\mu > 0$ large enough; while $4 < q < 6$ they obtained system (1.2) has at least a positive solution for all $\mu > 0$. Recently, Lei-Liu-Chu-Suo [9] considered the following Schrödinger-Poisson system

$$\begin{cases} -\Delta u + u + \eta \phi u = u^5 + \lambda f(x) u^{q-1}, & x \in \mathbb{R}^3, \\ -\Delta \phi = u^2, & x \in \mathbb{R}^3, \end{cases}$$

where $1 < q < 2, \eta \in \mathbb{R} \backslash \{0\}, \lambda > 0$ is a real parameter and $f \in L^{\frac{6}{6-q}}(\mathbb{R}^3)$ is a nonzero nonnegative function. Using the variational methods, they obtained that there exists a positive constant λ_* such that for all $\lambda \in (0, \lambda_*)$, the system has at least two positive solutions. A natural question is whether there exist solutions for the critical Schrödinger-Poisson system with nonhomogeneous term (that is, the case of $q = 1$ in system (1.2)). Ye [10] studied the following a class of nonhomogeneous Schrödinger-Poisson system

$$\begin{cases} -\Delta u + u + \lambda \phi u = f(u) + h(x), & x \in \mathbb{R}^3, \\ -\Delta \phi = u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.3}$$

where $\lambda > 0$ is a parameter and $0 \leq h(x) = h(|x|) \in L^2(\mathbb{R}^3), f$ satisfies the following hypotheses:

(f_1) $f \in C(\mathbb{R}, \mathbb{R}^+), f(0) = 0, f(t) \equiv 0$ for $t < 0$ and there exist $a > 0$ and $q \in (2,6)$ such that

$$f(t) \leq a(1 + |t|^{q-1}), \qquad \forall t \in \mathbb{R}.$$

(f_2) $\lim_{t \to 0} \frac{f(t)}{t} = 0$.

(f_3) $\lim_{t \to \infty} \frac{f(t)}{t} = +\infty$.

She proved that system (1.3) has at least two positive solutions with the aid of Ekeland's variational principle, Jeanjean's monotone method, Pohožaev's identity and the mountain pass theorem. However, the author did not consider the case of the critical exponent. Indeed, when the nonlinear term contains the critical exponential term, it is more difficult to study system (1.3).

The general form of the Schrödinger-Poisson system with critical exponent is as follows

$$\begin{cases} -\Delta u+u+l(x)\phi u=u^5+g(x,u), & x\in\mathbb{R}^3, \\ -\Delta\phi=l(x)u^2, & x\in\mathbb{R}^3, \end{cases} \tag{1.4}$$

where $l(x)$ and $g(x,s)$ satisfy some certain conditions. The system (1.4) has been extensively studied, for examples: [11-24]. Particularly, when $g(x,u)$ is superlinear, Huang and Rocha [13] studied system (1.4) in case of $g(x,u)=\mu h(x)|u|^{q-2}u$ with $2\le q<6$, and established a positive solution by using the variational methods. Zhang [21] studied the Schrödinger-Poisson system with a general nonlinearity in the critical growth. [22,23] investigated the ground state sign-changing solutions for the Schrödinger-Poisson system with critical growth.

Our paper is mainly inspired by [9,10]. Up to now, there was no information about system (1.1). Therefore, in this paper, we will study the existence of multiple solutions of system (1.1) with $\eta\neq 0$ by using the Mountain Pass theorem and variational method.

Our main result can be described as follows.

Theorem 1.1. *Assume that $\eta\neq 0$ and $f\in L^{\frac{6}{5}}(\mathbb{R}^3), f\ge 0, f\not\equiv 0$, then*
(i) when $\eta<0$, there exists a positive constant Λ_0 such that for all $\lambda\in(0,\Lambda_0)$, system (1.1) *has at least one positive solution (u_*,ϕ_{u_*})* in $H^1(\mathbb{R}^3)\times D^{1,2}(\mathbb{R}^3)$;
(ii) while $\eta>0$, there exists a positive constant Λ such that for all $\lambda\in(0,\Lambda)$, system (1.1) *has at least two positive solutions (u_*,ϕ_{u_*}) and $(u_{**},\phi_{u_{**}})$* in $H^1(\mathbb{R}^3)\times D^{1,2}(\mathbb{R}^3)$.

Remark 1.1. Compared to [10], on the one hand, we consider the critical system; on the other hand, we also study the negative coefficient of the nonlocal term. Compared to [9], we study the case of q=1 and also obtain two positive solutions for system (1.1) with η>0. However, there exists a mistake in the fifth conclusion of Lemma 2.1 in [9], we only can obtain $F(u_n)=F(u_n-u)+F(u)+o_n(1)$. Here, we correct this flaw and the corresponding proof. But, for the case of $\eta<0$, we could not obtain the second solutions because of the lack of compactness for the nonlocal term $\phi_u u$ in $H^1(\mathbb{R}^3)$.

This paper is organized as follows. In Section 2, we present some notations and prove some useful preliminary lemmas which pave the way for getting two positive solutions. Then we give the proof of Theorem 1.1.

2 Proof of Theorem 1.1

Throughout this paper, we make use of the following notations:

- $|u|_s = \left(\int_{\mathbb{R}^3} |u|^s \mathrm{d}x\right)^{\frac{1}{s}}$ is the usual Lebesgue space $L^s(\mathbb{R}^3)$ norm.
- The norm of $H^1(\mathbb{R}^3)$ is denoted by

$$\|u\| = \left(\int_{\mathbb{R}^3} \left(|\nabla u|^2 + u^2\right) \mathrm{d}x\right)^{\frac{1}{2}}.$$

  H^{-1} is the dual space of H^1.
- $D^{1,2}(\mathbb{R}^3) = \{u \in L^{2^*}(\mathbb{R}^3) : |\nabla u| \in L^2(\mathbb{R}^3)\}$ with the inner product

$$\int_{\mathbb{R}^3} (\nabla u, \nabla v) \mathrm{d}x.$$

- B_r (respectively, ∂B_r) is the closed ball (respectively, the sphere) of center zero and radius r, i.e.,

$$B_r = \{u \in H^1(\mathbb{R}^3) : \|u\| \le r\}, \qquad \partial B_r = \{u \in H^1(\mathbb{R}^3) : \|u\| = r\}.$$

- C, C_i $(i = 1, 2, \ldots)$ denote various positive constants, which may vary from line to line.
- For each $p \in [2,6)$, by the Sobolev constants, we denote

$$S = \inf_{u \in D^{1,2}(\mathbb{R}^3) \setminus \{0\}} \frac{\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x}{|u|_6^2}; \qquad S_p := \inf_{u \in H^1(\mathbb{R}^3) \setminus \{0\}} \frac{\|u\|^2}{|u|_p^2}.$$

As we all known that system (1.1) can be reduced to a nonlinear Schrödinger equation with nonlocal term. Indeed, the Lax-Milgram theorem implies that for any $u \in H^1(\mathbb{R}^3)$, there exists a unique $\phi_u \in D^{1,2}(\mathbb{R}^3)$ such that

$$-\Delta \phi_u = u^2.$$

We substitute ϕ_u to the first equation of system (1.1), then system (1.1) can be transformed into the following equation

$$-\Delta u + u + \eta \phi_u u = u^5 + \lambda f(x), \qquad x \in \mathbb{R}^3. \tag{2.1}$$

In order to obtain positive solutions, the Euler functional of Eq. (2.1) can be defined by I: $H^1(\mathbb{R}^3) \to \mathbb{R}$ as the following

$$I(u) = \frac{1}{2}\|u\|^2 + \frac{1}{4}\eta \int_{\mathbb{R}^3} \phi_u (u^+)^2 \mathrm{d}x - \frac{1}{6}\int_{\mathbb{R}^3} (u^+)^6 \mathrm{d}x - \lambda \int_{\mathbb{R}^3} f(x) u \mathrm{d}x,$$

where $u^+ = \max\{u,0\}$. We can deduce that the functional I is of class C^1 and its critical points are weak solutions of Eq. (2.1). Moreover, we can obtain that

$$\langle I'(u),\varphi\rangle = \int_{\mathbb{R}^3}(\nabla u\cdot\nabla\varphi + u\varphi)\mathrm{d}x + \eta\int_{\mathbb{R}^3}\phi_u u^+\varphi\mathrm{d}x - \int_{\mathbb{R}^3}(u^+)^5\varphi\mathrm{d}x - \lambda\int_{\mathbb{R}^3}f(x)\varphi\mathrm{d}x$$

for any $\varphi\in H^1(\mathbb{R}^3)$. According to [8] or [13], we have the following conclusions.

Lemma 2.1. *For every $u\in H^1(\mathbb{R}^3)$, there exists a unique $\phi_u\in D^{1,2}(\mathbb{R}^3)$ solution of*

$$-\Delta\phi = u^2$$

and the following results hold
(1) $\|\phi_u\|^2 = \int_{\mathbb{R}^3}\phi_u u^2\mathrm{d}x$,
(2) $\phi_u\geq 0$, *moreover* $\phi_u>0$ *when* $u\neq 0$,
(3) $\int_{\mathbb{R}^3}\phi_u u^2\mathrm{d}x = \int_{\mathbb{R}^3}|\nabla\phi_u|^2\mathrm{d}x \leq C\|u\|^4$,
(4) $F: H^1(\mathbb{R}^3)\to\mathbb{R}$ *is well defined with* $F(u)=\int_{\mathbb{R}^3}\phi_u u^2\mathrm{d}x$, *assume that* $u_n\rightharpoonup u$ *in* $H^1(\mathbb{R}^3)$, *then* $\phi_{u_n}\rightharpoonup\phi_u$ *in* $H^1(\mathbb{R}^3)$,
(5) F *is* C^1 *and*

$$\langle F'(u),v\rangle = 4\int_{\mathbb{R}^3}\phi_u uv\mathrm{d}x,\qquad \forall v\in H^1(\mathbb{R}^3).$$

Lemma 2.2. *There exist $\Lambda_0,\rho>0$ such that for each $\lambda\in(0,\Lambda_0)$, then it holds*

$$d := \inf_{u\in B_\rho(0)} I(u) < 0 \quad\text{and}\quad I|_{\partial B_\rho(0)}>0. \tag{2.2}$$

Proof. When $\eta>0$, by the Sobolev and Hölder inequalities, we obtain

$$\begin{aligned} I(u) &= \frac{1}{2}\|u\|^2 + \frac{1}{4}\eta\int_{\mathbb{R}^3}\phi_u(u^+)^2\mathrm{d}x - \frac{1}{6}\int_{\mathbb{R}^3}(u^+)^6\mathrm{d}x - \lambda\int_{\mathbb{R}^3}f(x)u\mathrm{d}x \\ &\geq \frac{1}{2}\|u\|^2 - \frac{1}{6}\int_{\mathbb{R}^3}(u^+)^6\mathrm{d}x - \lambda\int_{\mathbb{R}^3}f(x)u\mathrm{d}x \\ &\geq \frac{1}{2}\|u\|^2 - \frac{1}{6S_6^3}\|u\|^6 - \lambda S_6^{-\frac{1}{2}}|f|_{\frac{6}{5}}\|u\| \\ &= \|u\|\left(\frac{1}{2}\|u\| - \frac{1}{6S_6^3}\|u\|^5 - \lambda S_6^{-\frac{1}{2}}|f|_{\frac{6}{5}}\right). \end{aligned} \tag{2.3}$$

Set $g(t)=\frac{1}{2}t-\frac{1}{6S_6^3}t^5$, we can easily calculate that there exists a positive constant $\rho_1=(\frac{3}{5}S_6^3)^{\frac{1}{4}}$ such that $\max_{t>0}g(t)=g(\rho_1)>0$. Let

$$\lambda_* = \frac{S_6^{\frac{1}{2}}}{2|f|_{\frac{6}{5}}}g(\rho_1),$$

we have

$$I|_{\|u\|=\rho_1} \geq \frac{g(\rho_1)}{2}\rho_1 = \alpha > 0 \qquad \text{for any } \lambda \in (0,\lambda_*).$$

When $\eta < 0$, by the Sobolev and Hölder inequalities, we have

$$\begin{aligned} I(u) &= \frac{1}{2}\|u\|^2 + \frac{1}{4}\eta\int_{\mathbb{R}^3}\phi_u(u^+)^2\mathrm{d}x - \frac{1}{6}\int_{\mathbb{R}^3}(u^+)^6\mathrm{d}x - \lambda\int_{\mathbb{R}^3}f(x)u\mathrm{d}x \\ &= \frac{1}{2}\|u\|^2 - \frac{-\eta}{4}\int_{\mathbb{R}^3}\phi_u(u^+)^2\mathrm{d}x - \frac{1}{6}\int_{\mathbb{R}^3}(u^+)^6\mathrm{d}x - \lambda\int_{\mathbb{R}^3}f(x)u\mathrm{d}x \\ &\geq \frac{1}{2}\|u\|^2 - C\|u\|^4 - \frac{1}{6S_6^3}\|u\|^6 - \lambda S_6^{-\frac{1}{2}}|f|_{\frac{6}{5}}\|u\| \\ &= \|u\|\left(\frac{1}{2}\|u\| - C\|u\|^3 - \frac{1}{6S_6^3}\|u\|^5 - \lambda S_6^{-\frac{1}{2}}|f|_{\frac{6}{5}}\right). \end{aligned}$$

Set $g(t) = \frac{1}{2}t - Ct^3 - \frac{1}{6S_6^3}t^5$, we see that there exists a constant $\rho_2 > 0$ such that $\max_{t>0} g(t) = g(\rho_2) > 0$. Let

$$\lambda_{**} = \frac{S_6^{\frac{1}{2}}}{2|f|_{\frac{6}{5}}}g(\rho_2),$$

we have

$$I|_{\|u\|=\rho_2} \geq \frac{g(\rho_2)}{2}\rho_2, \qquad \text{for any } \lambda \in (0,\lambda_{**}).$$

Thus, set $\Lambda_0 = \min\{\lambda_*,\lambda_{**}\}, \rho = \min\{\rho_1,\rho_2\}$, then it follows that there exists a positive constant $\alpha = \min\left\{\frac{g(\rho_1)}{2}\rho_1, \frac{g(\rho_2)}{2}\rho_2\right\}$ such that $I(u) \geq \alpha$ for all $\|u\| = \rho$.

Moreover, by (2.3), we know that $d = \inf_{u\in B_\rho(0)} I(u)$ is well defined. Furthermore, for any $u \in H^1(\mathbb{R}^3)$, one has

$$\lim_{t\to 0^+}\frac{I(tu)}{t} = -\lambda\int_{\mathbb{R}^3}f(x)u\mathrm{d}x.$$

Thus, there exists $\hat{u} > 0$ such that $\|\hat{u}\| < \rho$ and $I(\hat{u}) < 0$. Consequently, $d = \inf_{u\in B_\rho(0)} I(u) < 0$. The proof is complete. □

Theorem 2.1. *Suppose* $0 < \lambda < \Lambda_0$ *(*Λ_0 *is defined in Lemma* 2.2*). Then system* (1.1) *has a positive solution* $(u_*,\phi_{u_*}) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$ *satisfying* $I(u_*) < 0$.

Proof. By Lemma 2.2, there exist $\alpha > 0, \rho > 0$ such that when $\lambda \in (0,\Lambda_0)$, for any $\|u\| = \rho$, we have $I(u) \geq \alpha > 0$ and $d = \inf_{u\in B_\rho(0)} I(u) < 0$. There exists a minimization sequence $\{u_n\} \subset B_\rho(0)$. Since $\{u_n\} \subset B_\rho(0)$, it's easy to see that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$. So there exist a subsequence (still denoted by itself) and $u_* \in H^1(\mathbb{R}^3)$ such that

$$\begin{cases} u_n \rightharpoonup u_* & \text{in } H^1(\mathbb{R}^3), \\ u_n \rightharpoonup u_* & \text{in } L^q_{\mathrm{loc}}(\mathbb{R}^3) \quad (2 \leq q \leq 6), \\ u_n(x) \to u_*(x) & \text{a.e. in } \mathbb{R}^3. \end{cases} \tag{2.4}$$

Set $w_n = u_n - u_*$, so $w_n \rightharpoonup 0$ in $H^1(\mathbb{R}^3)$. The Brézis-Lieb Lemma ([25] or [26]) implies that

$$\begin{cases} \|u_n\|^2 = \|w_n\|^2 + \|u_*\|^2 + o_n(1), \\ \int_{\mathbb{R}^3} (u_n^+)^6 \mathrm{d}x = \int_{\mathbb{R}^3} (w_n^+)^6 \mathrm{d}x + \int_{\mathbb{R}^3} (u_*^+)^6 \mathrm{d}x + o_n(1), \\ \int_{\mathbb{R}^3} \phi_{u_n} (u_n^+)^2 \mathrm{d}x = \int_{\mathbb{R}^3} \phi_{w_n} (w_n^+)^2 \mathrm{d}x + \int_{\mathbb{R}^3} \phi_{u_*} (u_*^+)^2 \mathrm{d}x + o_n(1). \end{cases} \tag{2.5}$$

Since $u_n \rightharpoonup u_*$ in $L^6(\mathbb{R}^3)$ and $f \in L^{\frac{6}{5}}(\mathbb{R}^3)$, we have

$$\int_{\mathbb{R}^3} f(x) u_n \mathrm{d}x = \int_{\mathbb{R}^3} f(x) u_* \mathrm{d}x + o_n(1). \tag{2.6}$$

By (2.2), for an appropriate constant ρ, we can deduce that

$$\frac{1}{2}\|u_n\|^2 + \frac{\eta}{4}\int_{\mathbb{R}^3} \phi_{u_n} (u_n^+)^2 \mathrm{d}x - \frac{1}{6}\int_{\mathbb{R}^3} (u_n^+)^6 \mathrm{d}x \geq 0, \qquad \text{for } u_n \in B_\rho(0). \tag{2.7}$$

If $u_* = 0$, then $w_n = u_n$, which follows that $w_n \in B_\rho(0)$. If $u_* \neq 0$, we also get $w_n \in B_\rho(0)$ for n large sufficiently. From (2.7), one has

$$\frac{1}{2}\|w_n\|^2 + \frac{\eta}{4}\int_{\mathbb{R}^3} \phi_{w_n} (w_n^+)^2 \mathrm{d}x - \frac{1}{6}\int_{\mathbb{R}^3} (w_n^+)^6 \mathrm{d}x \geq 0. \tag{2.8}$$

Therefore, by Lemma 2.1, it follows from (2.4)-(2.6) and (2.8) that

$$\begin{aligned} d = I(u_n) + o_n(1) = I(u_*) + \frac{1}{2}\|w_n\|^2 - \frac{1}{6}\int_{\mathbb{R}^3} (w_n^+)^6 \mathrm{d}x \\ + \frac{\eta}{4}\int_{\mathbb{R}^3} \phi_{w_n} (w_n^+)^2 \mathrm{d}x + o_n(1) \geq I(u_*) + o_n(1) \end{aligned} \tag{2.9}$$

as $n \to \infty$. Since $B_\rho(0)$ is closed and convex, thus $u_* \in B_\rho(0)$, we obtain $d \leq I(u_*)$. Hence, combining with (2.9), one has $I(u_*) = d < 0$ and $u_* \not\equiv 0$. It follows that u_* is a local minimizer of I. Since the functional I is of class C^1, one has u_* is a critical point of I, that is, u_* is a nonzero solution of Eq. (2.1). Thus, one has $\langle I'(u_*), \varphi \rangle = 0$ for any $\varphi \in H^1(\mathbb{R}^3)$. Particularly, choosing $\varphi = u_*^- = \min\{u_*, 0\}$, we have

$$\int_{\mathbb{R}^3} (\nabla u_* \cdot \nabla u_*^- + u_* u_*^-) \mathrm{d}x + \eta \int_{\mathbb{R}^3} \phi_{u_*} u_*^+ u_*^- \mathrm{d}x - \int_{\mathbb{R}^3} (u_*^+)^5 u_*^- \mathrm{d}x - \lambda \int_{\mathbb{R}^3} f(x) u_*^- \mathrm{d}x = 0,$$

that is,

$$\|u_*^-\|^2 - \lambda \int_{\mathbb{R}^3} f(x) u_*^- \mathrm{d}x = 0$$

which implies that $u_*^- \equiv 0$. So u_* is a nonzero and nonnegative solution of Eq. (2.1). Consequently, by the strong maximum principle, we get $u_* > 0$. So u_* is a positive solution of Eq. (2.1) with $I(u_*) < 0$. Therefore, we can conclude that (u_*, ϕ_{u_*}) is a positive solution of system (1.1). This completes the proof of Theorem 2.1. □

Lemma 2.3. *Assume that $\eta>0$, then the functional I satisfies the $(PS)_c$ condition provided $c<\frac{1}{3}S^{\frac{3}{2}}-D\lambda^2$, where $D=\frac{9}{16}(|f|_{\frac{6}{5}}S_6^{-\frac{1}{2}})^2$.*

Proof. Let $\{u_n\}\subset H^1(\mathbb{R}^3)$ be a $(PS)_c$ sequence of I, that is,

$$I(u_n)\to c, \quad I'(u_n)\to 0, \qquad \text{as } n\to\infty. \tag{2.10}$$

We claim that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$. For n large enough and combining with (2.10), one gets that

$$\begin{aligned} c+1+o(\|u_n\|)\geq I(u_n)-\frac{1}{4}\langle I'(u_n),u_n\rangle &=\frac{1}{4}\|u_n\|^2+\frac{1}{12}\int_{\mathbb{R}^3}(u_n^+)^6\mathrm{d}x-\frac{3}{4}\lambda\int_{\mathbb{R}^3}f(x)u_n\mathrm{d}x\\ &\geq\frac{1}{4}\|u_n\|^2-\frac{3}{4}\lambda S_6^{-\frac{1}{2}}|f|_{\frac{6}{5}}\|u_n\|, \end{aligned}$$

which implies that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$. Going if necessary to a subsequence, still denoted by $\{u_n\}$ and there exists $v\in H^1(\mathbb{R}^3)$ such that $u_n\rightharpoonup v$ weakly in $H^1(\mathbb{R}^3)$ as $n\to\infty$. Set $w_n=u_n-v$, similar to Theorem 2.1, one has (2.5) holds for v. If $\|w_n\|^2\to 0$, then the conclusion holds. Otherwise, there exists a subsequence (still denoted by itself) such that $\lim_{n\to\infty}\|w_n\|^2=l>0$. From (2.10), for any $\varphi\in H^1(\mathbb{R}^3)$, we have $\langle I'(u_n),\varphi\rangle\to 0$. By Lemma 2.1 and (2.6), as $n\to\infty$, it follows that

$$\int_{\mathbb{R}^3}(\nabla v\cdot\nabla\varphi+v\varphi)\mathrm{d}x+\eta\int_{\mathbb{R}^3}\phi_v(x)v^+\varphi\mathrm{d}x-\int_{\mathbb{R}^3}(v^+)^5\varphi\mathrm{d}x-\lambda\int_{\mathbb{R}^3}f(x)\varphi\mathrm{d}x=0. \tag{2.11}$$

Taking the test function $\varphi=v$ in (2.11), then it holds that

$$\|v\|^2+\eta\int_{\mathbb{R}^3}\phi_v(v^+)^2\mathrm{d}x-\int_{\mathbb{R}^3}(v^+)^6\mathrm{d}x-\lambda\int_{\mathbb{R}^3}f(x)v\mathrm{d}x=0. \tag{2.12}$$

From (2.10), we have $\langle I'(u_n),u_n\rangle\to 0$. By Lemma 2.1, (2.5) and (2.6), we obtain

$$\begin{aligned} o_n(1)=&\|v\|^2+\eta\int_{\mathbb{R}^3}\phi_v(v^+)^2\mathrm{d}x+\eta\int_{\mathbb{R}^3}\phi_{w_n}(w_n^+)^2\mathrm{d}x-\int_{\mathbb{R}^3}(v^+)^6\mathrm{d}x\\ &+\|w_n\|^2-\int_{\mathbb{R}^3}(w_n^+)^6\mathrm{d}x-\lambda\int_{\mathbb{R}^3}f(x)v\mathrm{d}x. \end{aligned} \tag{2.13}$$

It follows from (2.12) and (2.13) that

$$\|w_n\|^2+\eta\int_{\mathbb{R}^3}\phi_{w_n}(w_n^+)^2\mathrm{d}x-\int_{\mathbb{R}^3}(w_n^+)^6\mathrm{d}x=o_n(1). \tag{2.14}$$

By the Sobolev inequality, since $\eta>0$, we have

$$|w_n^+|_6^2\leq|w_n|_6^2\leq S^{-1}\int_{\mathbb{R}^3}|\nabla w_n|^2\mathrm{d}x\leq S^{-1}\|w_n\|^2.$$

Consequently, we can obtain $l\geq S^{\frac{3}{2}}$.

On the one hand, by (2.12), the Hölder inequality, Young inequality and Sobolev inequality, it holds that

$$\begin{aligned}
I(v)=&\frac{1}{2}\|v\|^2+\frac{1}{4}\eta\int_{\mathbb{R}^3}\phi_v(v^+)^2\mathrm{d}x-\frac{1}{6}\int_{\mathbb{R}^3}(v^+)^6\mathrm{d}x-\lambda\int_{\mathbb{R}^3}f(x)v\mathrm{d}x\\
=&\frac{1}{4}\|v\|^2+\frac{1}{12}\int_{\mathbb{R}^3}(v^+)^6\mathrm{d}x-\frac{3}{4}\lambda\int_{\mathbb{R}^3}f(x)v\mathrm{d}x\\
\geq&\frac{1}{4}\|v\|^2-\frac{3}{4}\lambda S_6^{-\frac{1}{2}}|f|_{\frac{6}{5}}\|v\|\\
\geq&\frac{1}{4}\|v\|^2-\left[\frac{1}{4}\|v\|^2+\frac{9}{16}\left(\lambda|f|_{\frac{6}{5}}S_6^{-\frac{1}{2}}\right)^2\right]\\
\geq&\frac{1}{4}\|v\|^2-\frac{1}{4}\|v\|^2-\frac{9}{16}\left(\lambda|f|_{\frac{6}{5}}S_6^{-\frac{1}{2}}\right)^2\\
=&-D\lambda^2,
\end{aligned}\tag{2.15}$$

where $D=\frac{9}{16}|f|_{\frac{6}{5}}^2S_6^{-1}$.

On the other hand, when $\eta>0$, it follows from (2.6),(2.10) and (2.14) that

$$\begin{aligned}
I(v)=&I(u_n)-\frac{1}{2}\|w_n\|^2+\frac{1}{6}\int_{\mathbb{R}^3}(w_n^+)^6\mathrm{d}x-\frac{\eta}{4}\int_{\mathbb{R}^3}\phi_{w_n}(w_n^+)^2\mathrm{d}x+o_n(1)\\
\leq&I(u_n)-\frac{1}{2}\|w_n\|^2+\frac{1}{6}\int_{\mathbb{R}^3}(w_n^+)^6\mathrm{d}x-\frac{\eta}{6}\int_{\mathbb{R}^3}\phi_{w_n}(w_n^+)^2\mathrm{d}x+o_n(1)\\
=&I(u_n)-\frac{1}{3}\|w_n\|^2+o_n(1)\\
=&c-\frac{1}{3}l+o_n(1)\\
<&c-\frac{1}{3}S^{\frac{3}{2}}\\
<&-D\lambda^2,
\end{aligned}$$

which contradicts (2.15). Therefore $l=0$. The proof is complete. $\square$

We know that the extremal function

$$U(x)=\frac{(3\varepsilon^2)^{\frac{1}{4}}}{(\varepsilon^2+|x|^2)^{\frac{1}{2}}},\qquad x\in\mathbb{R}^3$$

solves

$$-\Delta u=u^5\qquad\text{in }\mathbb{R}^3\backslash\{0\}$$

and $|\nabla U|_2^2=|U|_6^6=S^{\frac{3}{2}}$. We choose a function $\zeta\in C_0^\infty(\mathbb{R}^3)$ such that $0\leq\zeta(x)\leq 1$ in $\mathbb{R}^3$. $\zeta(x)=1$ near $x=0$ and it is radially symmetric. We define

$$u_\varepsilon(x)=\zeta(x)U(x).$$

Besides, since (u_*,ϕ_{u_*}) is a positive solution of system (1.1), by a standard method, we can obtain that there exist $m,M>0$ such that $m\le u_*\le M$ for each $x\in \mathrm{supp}\zeta$.

Lemma 2.4. *Assume that $\eta>0$ and $f\in L^{\frac{6}{5}}(\mathbb{R}^3), f\ge 0, f\not\equiv 0$, then there exist $\Lambda_1>0$ and $u_\varepsilon\in H^1(\mathbb{R}^3)$ such that*

$$\sup_{t\ge 0} I(u_*+tu_\varepsilon)<\frac{1}{3}S^{\frac{3}{2}}-D\lambda^2,\qquad \text{for all } \lambda\in(0,\Lambda_1).$$

Proof. From [27], one has

$$|u_\varepsilon|_6^6=|U|_6^6+O(\varepsilon^3)=S^{\frac{3}{2}}+O(\varepsilon^3),\qquad \|u_\varepsilon\|^2=|\nabla U|_2^2+O(\varepsilon)=S^{\frac{3}{2}}+O(\varepsilon), \tag{2.16}$$

$$|u_\varepsilon|_p^p=\begin{cases} O(\varepsilon^{\frac{p}{2}}), & 1\le p<3,\\ O(\varepsilon^{\frac{p}{2}}|\ln\varepsilon|), & p=3,\\ O(\varepsilon^{3-\frac{p}{2}}), & p>3. \end{cases} \tag{2.17}$$

It is obvious that the following inequality

$$(a+b)^6\ge a^6+b^6+6a^5b+6ab^5$$

holds for each $a,b\ge 0$. Since u_* is a positive solution of Eq. (2.1) with $I_\lambda(u_*)<0$, by the above inequality, for all $t\ge 0$ we have

$$\begin{aligned} I(u_*+tu_\varepsilon)=&I(u_*)+\frac{1}{2}t^2\|u_\varepsilon\|^2+t\int_{\mathbb{R}^3}\left[\nabla u_*\cdot\nabla u_\varepsilon+u_*u_\varepsilon+\eta\phi_{u_*}u_*u_\varepsilon-u_*^5u_\varepsilon-\lambda f(x)u_\varepsilon\right]\mathrm{d}x\\ &+\frac{1}{4}\eta\int_{\mathbb{R}^3}\left[\phi_{u_*+tu_\varepsilon}(u_*+tu_\varepsilon)^2-\phi_{u_*}u_*^2-4t\phi_{u_*}u_*u_\varepsilon\right]\mathrm{d}x\\ &-\frac{1}{6}\int_{\mathbb{R}^3}[(u_*+tu_\varepsilon)^6-u_*^6-6tu_*^5u_\varepsilon]\mathrm{d}x\\ \le&\frac{1}{2}t^2\|u_\varepsilon\|^2-\frac{1}{6}t^6\int_{\mathbb{R}^3}u_\varepsilon^6\mathrm{d}x-t^5\int_{\mathbb{R}^3}u_*u_\varepsilon^5\mathrm{d}x+g_\varepsilon(t)\\ \le&\frac{1}{2}t^2\|u_\varepsilon\|^2-\frac{1}{6}t^6\int_{\mathbb{R}^3}u_\varepsilon^6\mathrm{d}x-mt^5\int_{\mathbb{R}^3}u_\varepsilon^5\mathrm{d}x+g_\varepsilon(t), \end{aligned} \tag{2.18}$$

where

$$g_\varepsilon(t)=\frac{1}{4}\eta\int_{\mathbb{R}^3}\left[\phi_{u_*+tu_\varepsilon}(u_*+tu_\varepsilon)^2-\phi_{u_*}u_*^2-4t\phi_{u_*}u_*u_\varepsilon\right]\mathrm{d}x.$$

According to [9], we can get that

$$g_\varepsilon(t)\le Ct^2\varepsilon+Ct^3\varepsilon^{\frac{3}{2}}+Ct^4\varepsilon^2.$$

Set

$$h_\varepsilon(t)=\frac{1}{2}t^2\|u_\varepsilon\|^2-\frac{1}{6}t^6\int_{\mathbb{R}^3}u_\varepsilon^6\mathrm{d}x-t^5m\int_{\mathbb{R}^3}u_\varepsilon^5\mathrm{d}x+Ct^2\varepsilon+Ct^3\varepsilon^{\frac{3}{2}}+Ct^4\varepsilon^2.$$

Since $\lim_{t\to+\infty} h_\varepsilon(t) = -\infty$ and $h_\varepsilon(0)=0$, there exist $t_1, t_2 > 0$ such that

$$0 < t_1 \le t_\varepsilon \le t_2 < \infty \tag{2.19}$$

and

$$h_\varepsilon(t_\varepsilon) = \sup_{t\ge 0} h_\varepsilon(t), \qquad h'_\varepsilon(t)|_{t=t_\varepsilon} = 0.$$

By (2.17), one has

$$\int_{\mathbb{R}^3} u_\varepsilon^5 \mathrm{d}x = O(\varepsilon^{\frac{1}{2}}).$$

Consequently, it follows from (2.16) and (2.19) that

$$\sup_{t\ge 0} h_\varepsilon(t) \le \sup_{t\ge 0} \left\{ \frac{1}{2} t^2 S^{\frac{3}{2}} - \frac{1}{6} t^6 S^{\frac{3}{2}} \right\} + C_1\varepsilon - C_2 \varepsilon^{\frac{1}{2}} \le \frac{1}{3} S^{\frac{3}{2}} + C_1 \varepsilon - C_2 \varepsilon^{\frac{1}{2}}, \tag{2.20}$$

where $C_1, C_2 > 0$ (independent of ε, λ). Let $\varepsilon = \lambda^2$, $0 < \lambda < \frac{C_2}{C_1 + D}$, then we have that

$$C_1 \varepsilon - C_2 \varepsilon^{\frac{1}{2}} = C_1 \lambda^2 - C_2 \lambda = \lambda^2 (C_1 - C_2 \lambda^{-1}) < -D\lambda^2.$$

Consequently, combining with (2.20), one has

$$\sup_{t\ge 0} I(u_* + t u_\varepsilon) \le \sup_{t \ge 0} h_\varepsilon(t) < \frac{1}{3} S^{\frac{3}{2}} - D\lambda^2.$$

This completes the proof of Lemma 2.4. □

Theorem 2.2. *Assume that $\eta > 0$ and $f \in L^{\frac{6}{5}}(\mathbb{R}^3), f \ge 0, f \not\equiv 0$, then system* (1.1) *has another positive solution $(u_{**}, \phi_{u_{**}})$ with $I(u_{**}) > 0$.*

Proof. Let $\Lambda = \min\{\Lambda_0, \Lambda_1, (\frac{S^{\frac{2}{3}}}{3D})^{\frac{1}{2}}\}$. By Lemma 2.4, we can choose a sufficiently large $T_0 > 0$ such that $I(u_* + T_0 u_\varepsilon) < 0$, with the fact that $I(u_*) < 0$. Then, applying the Mountain-pass Lemma (see [28]), we obtain that there exists a sequence $\{u_n\} \subset H^1(\mathbb{R}^3)$ such that

$$I(u_n) \to c > 0 \quad \text{and} \quad I'(u_n) \to 0,$$

where

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t))$$

and

$$\Gamma = \{\gamma \in C([0,1], H^1(\mathbb{R}^3)) \mid \gamma(0) = u_*, \gamma(1) = u_* + T_0 u_\varepsilon\}.$$

By Lemma 2.3, there exists a convergent subsequence $\{u_n\}$ (still denoted by $\{u_n\}$) and $u_{**} \in H^1(\mathbb{R}^3)$ such that $u_n \to u_{**}$ in $H^1(\mathbb{R}^3)$, thus u_{**} is a solution of Eq. (2.1) with $I(u_{**}) > 0$. Similar to u_*, we can also get $u_{**} \ge 0$ and $u_{**} \not\equiv 0$. By using the strong maximum principle, we have $u_{**} > 0$ in $\mathbb{R}^3$. Thus, $(u_{**}, \phi_{u_{**}})$ is a positive solution of system (1.1). The proof of Theorem 2.2 is completed. □

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