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Well-Posedness and Stability for Semilinear Thermoelastic System with Boundary Time-Varying Delay and Nonlinear Weight

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Abstract. This paper is concerned with the well-posedness, uniform asymptotic stability and dynamics for a semilinear thermoelastic system with time-varying delay boundary feedback and nonlinear weight, which can be used to describe the physical procedure of meridian retraction and release therapy. The perturbation theory of linear operators by Kato is used to deal with the invalidity of Lumper-Phillips theorem on non-autonomous PDEs operator, the multiplier approach and quasi-stability method lead to the stability and dynamics for our semilinear problem, which are also true for linear thermoelastic system without weight.

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1 Introduction

The delay and memory influence the stability and dynamics for evolutionary differential equations, which come from physics, biology, medicine, material, artificial intelligence and applied science/engineer, such as the transmission problem of hyperbolic equations.

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In the traditional Chinese medicine, there exists a physical therapy procedure which is called as meridian retraction and release therapy. The principle of this therapy can be seen as an abstract and simplified mathematical model influenced by delay, which is consisted by a coupled system via wave propagation and heat transportation with time-varying delay boundary feedback and nonlinear weight as the following semilinear thermoelastic system

$$\begin{cases} au_{tt}-du_{xx}+\beta\theta_x+h_1(u)=0, & \text{in } (0,L)\times(0,\infty), \\ b\theta_t-\kappa\theta_{xx}+\beta u_{xt}+h_2(\theta)=0, & \text{in } (0,L)\times(0,\infty), \\ u_x(0,t)=u(L,t)=\theta(0,t)=0, & t\geq 0, \\ \theta_x(L,t)+k_1(t)\theta(L,t)+k_2(t)\theta(L,t-\tau(t))=0, & t\geq 0, \\ u(x,0)=u_0(x),\ u_t(x,0)=u_1(x),\ \theta(x,0)=\theta_0(x), & t\geq 0, \\ \theta(L,t-\tau(0))=f_0(L,t-\tau(0)), & \text{in } (0,L)\times(0,\tau(0)), \end{cases} \tag{1.1}$$

where (u_0,u_1,θ_0,f_0) belongs to some appropriate Sobolev space, $u(t,x)$ is the displacement of wave along meridians, θ denotes the heat flow which obeys the Fourier law, $h_i(\cdot)$ $(i=1,2)$ are the semilinear external forces caused by the retraction and release therapy technique on the boundary point $x=L$. Since the stability and dynamics are important in curative effect from the view of theory, the discussion of problem (1.1) is our objective in this paper.

When the semilinear terms $h_1(u)$ and $h_2(\theta)$ equal to 0, the weights $k_1(t)$ and $k_2(t)$ reduce to constants, the system (1.1) is degraded into the problem in [1]. Especially, if the time-varying delay $\tau(t)$ becomes constant, (1.1) reduces to the problem in [2].

As our best acknowledge, the research on hyperbolic equation with delay has been investigated in fruitful literatures, for instance [3–13]. The thermoelastic systems contain three types according to the different damping, which are composed by wave equation and heat flow. The well-posedness and stability for classical thermoelastic system have been studied in last thirty years, which can be referred in monographs as Jiang and Racke [14], Liu and Zheng [15]. The controllability and stability for thermoelastic system can be seen in [16, 17] and some related literatures therein. Originated from the idea in Nicaise, Pignotti and Valein [8], Mustafa [1, 2] considered the thermoelastic systems with boundary feedbacks which contain constant, time-varying and distributed delays, and derived the stability of energy functional. Thereafter, Mustafa and his collaborator Kafini, Messaoudi in [18–20] investigated the Timoshenko-type system of thermoelasticity with delays and showed the energy decay. Based on the above related literatures, the global well-posedness, stability and dynamics for semilinear thermoelastic system with time-varying delay and nonlinear weight are our objective in this presented paper, which contains the following results and features.

(1) The problem (1.1) is using to describe the transportation in medicine here, the global well-posedness has been achieved by semigroup theory in [21] together with the

perturbation of linear operators by Kato in [19]. By virtue of the Dafermos transformation, (1.1) can be written as an equivalent abstract form (2.25), the difficulty here is the operator $\mathcal{A}(t)$ is non-autonomous, which leads to the invalidity of Lumper-Phillips Theorem as in [15].

By introducing the perturbation theory of non-autonomous linear operator as in Theorem 2.1 from Kato [19], we can derive the desired well-posedness for (1.1) via verification Theorem 2.1 of perturbed operator $\tilde{A}(t)$ in Subsection 2.5, which presents a strict proof for existence of global weak solution. The procedure in Subsection 2.5 also gives strict formal analysis for the linear case of problem (1.1) without weight in Mustafa [1].

(2) Using multiplier technique for energy functional of (1.1), the quasi-stability and uniform asymptotic stability of the gradient system for (1.1) have been attained. From the quasi-stability method introduced by Chueshov and Lasiecka [22, 23], Chueshov [24], the finite dimensional global and exponential attractors $\mathcal{A}$ and $\mathcal{A}^{exp}$ have been shown respectively, here $\mathcal{A}$ is consisted by the unstable manifold $\mathbb{M}^u(\mathcal{N})$ with the set of stationary points $\mathcal{N}$. Our results of stability and dynamics are the generation of [1] and [2].

(3) Since the balance between nonlinear weights $k_1(t)$ and $k_2(t)$ in the boundary feedback is crucial for the stability of gradient system, the hypotheses (H1) and (H2) guarantee the asymptotic stability of gradient system, which are sufficient conditions. Otherwise, the instability similar as in Nicaise and Pignotti [7] can be obtained or not, this is still unknown.

The rest of this article is arranged as follows: In Section 2, we present some preliminaries and global well-posedness to (1.1). The stability and dynamics for gradient system have been shown in Section 3.

2 Global well-posedness

2.1 Some useful lemmas and remarks

Consider the following Cauchy problem

$$\begin{cases} \dfrac{\mathrm{d}U}{\mathrm{d}t} = A(t)U, \\ U(0) = U_0. \end{cases} \tag{2.1}$$

Then the global well-posedness of problem (2.1) is derived by the perturbation theory of linear operator from Kato [25] as following theorem.

Theorem 2.1 ([25]). *Assume that*
(1) $Y = D(A(0))$ *is a dense subset of* $\mathcal{H}$.
(2) $D(A(t))$ *is independent on time* t, $D(A(t)) = D(A(0))$ *for all* $t > 0$.
(3) $A(t)$ *generates a strongly continuous semigroup on* $\mathcal{H}$ *for all* $t \in [0,T]$, *and the family* $A = \{A(t) | t \in [0,T]\}$ *is stable with stability constants* C *and* m *independent on* t, *i.e., the semigroup* $(S_t(s))_{s \geq 0}$ *generated by* $A(t)$ *which satisfies*

$$\|S_t(s)U\|_{\mathcal{H}} \leq Ce^{ms}\|U\|_{\mathcal{H}},$$

for all $U \in \mathcal{H}$ *and* $s \geq 0$.
(4) $A_t(t)$ *belongs to* $L^\infty_*([0,T], B(Y,\mathcal{H}))$, *which is the space of equivalent classes of essentially bounded, strongly measurable functions form* $[0,T]$ *into the set* $B(Y,\mathcal{H})$ *of bounded linear operators from* Y *into* $\mathcal{H}$.

Then, problem (2.1) *possesses a unique solution* $U \in C([0,T],Y) \cap C^1([0,T],\mathcal{H})$ *for any initial data in* Y.

Next, we will present some remarks for the preparation of discussion in sequel.

Remark 2.1. Since problem (1.1) is defined in one dimension, although the boundary contains Neumann case, the Poincaré inequality still holds, this is important in the process of multiplier approach, see [26].

Remark 2.2. The estimate of θ satisfies

$$\frac{1}{L}\theta^2(L,t) \leq \Big(\int_0^L \theta_x \mathrm{d}x\Big)^2 \leq \int_0^L \theta^2 \mathrm{d}x. \tag{2.2}$$

2.2 The abstract theory of non-autonomous hyperbolic-type evolutionary equations

This section is to present local existence of mild solution for preparation, which can be found in [27,28] and [21].

Denote $\Lambda(t) = -A(t)$, let $\mathcal{Z}$ as a abstract Banach space, consider the Cauchy problem for non-autonomous hyperbolic-type evolutionary equations as

$$\begin{cases} \dfrac{\mathrm{d}U}{\mathrm{d}t} + \Lambda(t)U = f(t,U), \\ U(\tau) = U_\tau \in \mathcal{Z}, \end{cases} \tag{2.3}$$

where the operator $\Lambda(t)$ satisfies

(i) $\{\Lambda(t)\}$ is non-negative stable generator of C_0-semigroup in $\mathcal{Z}$ with stable constants M and ω, where the stability is defined provided that there exist constant $M \geq 1$ and stable parameter ω such that

$$(\omega, +\infty) \subset \rho(\Lambda(t)), \qquad t \in [\tau, T] \tag{2.4}$$

and
$$\Big\|\prod_{j=1}^{k}(\lambda+\Lambda(t_j))^{-1}\Big\|_{\mathcal{L}(\mathcal{Z})}\le M(\lambda-\omega)^{-k},\qquad \lambda>\omega \tag{2.5}$$
for $\tau\le t_1\le t_2\le\cdots\le t_k\le T$ with $k=1,2,\cdots$, $\rho(\cdot)$ denotes the resolvent set.

(ii) Let $\mathcal{Y}$ be a Banach space satisfies $Y\hookrightarrow\mathcal{Z}$ and $\mathcal{Y}$ is dense in $\mathcal{Z}$. Then the domain $D(\Lambda(t))=\mathcal{Y}$, where $\mathcal{Y}$ is independent on the choice of t.

(iii) $\Lambda(t)\in\mathrm{Lip}_*([\tau,+\infty);\mathcal{L}(\mathcal{Y},\mathcal{Z}))$ or $\partial_t\Lambda(t)\in L^\infty_*([\tau,+\infty);\mathcal{L}(\mathcal{Y},\mathcal{Z}))$, more details can be seen in [27].

In addition, the nonlinear term $f(\cdot,\cdot):\mathbb{R}\times\mathcal{Z}\to\mathcal{Z}$ satisfies that for all $R>0$, there exists $C=C(R)>0$ such that
$$\begin{cases}\|f(t,U)-f(t,V)\|_{\mathcal{Z}}\le C(R)\|U-V\|_{\mathcal{Z}},\\ \|f(t,U)\|_{\mathcal{Z}}\le C(R).\end{cases} \tag{2.6}$$

Theorem 2.1 ([21,28])**.** Assume the hypotheses (i)–(iii) hold. Then there exists a unique linear evolutionary process $S(t,s)$ with $\tau\le s\le t$ in $\mathcal{Z}$ satisfies

(a) $S:[\tau,\infty)\to\mathcal{L}(\mathcal{Z})$ is strong continuous and $S(s,s)=Id$.

(b) $S(t,s)S(s,r)=S(t,r)$ for all $r\le s\le t$.

(c) $S(t,s)\mathcal{Y}\subset\mathcal{Y}$ and $S:[\tau,\infty)\to\mathcal{L}(\mathcal{Y})$.

(d) $\frac{\mathrm{d}S(t,s)}{\mathrm{d}t}=-\Lambda(t)S(t,s)$ holds in the strong topology of $\mathcal{L}(\mathcal{Y},\mathcal{Z})$.

(e) $\frac{\mathrm{d}S(t,s)}{\mathrm{d}s}=S(t,s)\Lambda(s)$, which is strong continuous from $[\tau,\infty)$ to $\mathcal{L}(\mathcal{Y},\mathcal{Z})$.

(f) $\|S\|_{\infty,\mathcal{Z}}:=\sup_{t,s\in[\tau,\infty)}\|S(t,s)\|_{\mathcal{L}(\mathcal{Z})}\le Me^{\omega(t-s)}$.

Definition 2.1. *The function $U:[\tau,\tau+\tau_0]\to\mathcal{Z}$ is a mild solution for problem* (2.3) *for appropriate τ_0 provided that $U\in C([\tau,\tau+\tau_0],\mathcal{Z})$ satisfying*
$$U(t)=S(t,\tau)u_\tau+\int_\tau^t S(t,x)f(s,U(s))\mathrm{d}s. \tag{2.7}$$

Theorem 2.2 (Local existence of mild solution–See [27])**.** *Assume the hypotheses (i)–(iii) hold, the nonlinear term f satisfies* (2.6)*. Then for arbitrary $U_\tau\in B_{\mathcal{Z}}(0,r)$, the abstract problem* (2.3) *possesses a unique mild solution $U(\cdot;\tau,U_\tau):[\tau,\tau+\tau_0]\to\mathcal{Z}$ which is continuously dependent on the initial data for appropriate $\tau_0>0$.*

2.3 Assumptions

For the well-posedness and stability, the forcing assumptions on $k_1(t)$ and $k_2(t)$ are stated as follows.
(H1): The function $k_1(t):\mathbb{R}_+\to(0,\infty)$ is non-increasing and belongs to $C^1(\mathbb{R}_+)$, which satisfies

$$\Big|\frac{k_1'(t)}{k_2(t)}\Big|\le M_1 \tag{2.8}$$

for all $t\ge 0$ and some constant $M_1>0$.
(H2): The function $k_2(t):\mathbb{R}_+\to\mathbb{R}$ is a $C^1(\mathbb{R}_+)$-class function such that

$$|k_2(t)|\le \alpha k_1(t) \tag{2.9}$$
$$|k_1'(t)|\le M_1 k_1(t) \tag{2.10}$$

hold for some $0<\alpha<\frac{2}{a}$ and $M_2>0$.
(H3): For $h_i(\cdot)\in C^2(\Omega)$ with $i=1,2$, there exist constants $C_{h_i}>0$ such that

$$|h_i''(s)|\le C_{h_i}(1+|s|),\qquad \text{for all } s\in\mathbb{R}, \tag{2.11}$$

which implies that, for some $C_{h_i}>0$,

$$|h_i(s_1)-h_i(s_2)|\le C_{h_i}(1+|s_1|^2+|s_2|^2)|s_1-s_2|,\qquad \text{for all } s_1,s_2\in\mathbb{R}. \tag{2.12}$$

In addition, assume that there exist $\rho_f>0$, $\ell_0=\min\{\frac{d}{4},\frac{\kappa}{4}\}>0$, such that

$$h_i(s)s\ge -\ell_0\tilde{\Lambda}s^2-\rho_{h_i}\quad\text{and}\quad H_i(s)\ge -\frac{\ell_0\tilde{\Lambda}}{2}s^2-\rho_{h_i},\qquad \text{for all } s\in\mathbb{R} \tag{2.13}$$

and

$$\liminf_{|s|\to\infty} h_i'(s)>-\ell_0\tilde{\Lambda} \tag{2.14}$$

with $H_i(s)=\int_0^s h_i(\sigma)\mathrm{d}\sigma$ hold, where $\tilde{\Lambda}$ is the constant in the Poincaré inequality for our one dimensional problem.
(H4): Moreover, similar as in [6,8,11], we assume that

$$\tau(t)\in W^{2,\infty}([0,T]),\qquad \forall T>0 \tag{2.15}$$
$$0<\tau_0\le\tau(t)\le\tau_1,\qquad |\tau'(t)|\le l<1 \tag{2.16}$$
$$|\tau''(t)|\le\tilde{l} \tag{2.17}$$

for some positive parameters τ_0, τ_1, d and all $t>0$.
(H5): The parameters satisfy $\dfrac{\beta}{2d\tilde{\Lambda}}\le\dfrac{\kappa}{4}$.

2.4 The equivalent system

Originated from Dafermos [29], and Nicaise and Pignotti [11], we introduce the Dafermos transformation

$$z(\rho,t)=\theta(L,t-\rho\tau(t)) \tag{2.18}$$

for $(\rho,t)\in(0,1)\times(0,\infty)$, then the problem (1.1) is equivalent to

$$\begin{cases} au_{tt}-du_{xx}+\beta\theta_x+h_1(u)=0, & \text{in } (0,L)\times(0,\infty),\\ b\theta_t-\kappa\theta_{xx}+\beta u_{xt}+h_2(\theta)=0, & \text{in } (0,L)\times(0,\infty),\\ \tau(t)z_t(\rho,t)+(1-\tau'(t)\rho)z_\rho(\rho,t)=0, & \\ u_x(0,t)=u(L,t)=\theta(0,t)=0, & t\geq 0,\\ \theta_x(L,t)+k_1(t)z(0,t)+k_2(t)z(1,t)=0, & t\geq 0,\\ u(x,0)=u_0(x),\ u_t(x,0)=u_1(x),\ \theta(x,0)=\theta_0(x), & t\geq 0,\\ \theta(L,t-\tau(0))=f_0(L,t-\tau(0)), & \text{in } (0,L)\times(0,\tau(0)) \end{cases} \tag{2.19}$$

for $z(0,t)=\theta(L,t)$ and $z(1,t)=\theta(L,t-\tau(t))$.

2.5 Global well-posedness

In this part, the existence and uniqueness of global solution will be shown by using Kato's operator perturbed theory and semigroup theory.

The phase space $\mathcal{H}$ is Hilbert space, which is defined

$$\mathcal{H}=V_1\times L^2(0,L)\times L^2(0,L)\times L^2((0,1)\times\mathbb{R}_+) \tag{2.20}$$

with

$$V_1=\{w\in H^1(0,L)|w(L)=0\}, \tag{2.21}$$
$$V_2=\{w\in H^1(0,L)|w(0)=0\}, \tag{2.22}$$
$$V_3=\{w\in H^2(0,L)|w_x(0)=w(L)=0\}, \tag{2.23}$$

and inner product of $\mathcal{H}$ is equipped with

$$(U,V)_{\mathcal{H}}=\int_0^L(du_x\bar{u}_x+av\bar{v}+b\theta\bar{\theta})\mathrm{d}x+\xi(t)\tau(t)\int_0^1 z(\rho,t)\bar{z}(\rho,t)\mathrm{d}\rho \tag{2.24}$$

for $U=(u,v,\theta,z)^T$ and $V=(\bar{u},\bar{v},\bar{\theta},\bar{z})$ in $\mathcal{H}$.

Denoting $v=u_t$ and $U=(u,v,\theta,z)^T$, then the problem can be rewritten as

$$\begin{cases} \dfrac{\mathrm{d}U}{\mathrm{d}t}-\mathcal{A}(t)U=H(u,\theta),\\ U(0)=U_0=(u_0,u_1,\theta_0,f_0), \end{cases} \tag{2.25}$$

where $H(\cdot,\cdot)$ is defined as

$$H(u,\theta)=\begin{pmatrix} 0 \\ -h_1(u) \\ -h_2(\theta) \\ 0 \end{pmatrix}$$

and the operator $\mathcal{A}(t):D(\mathcal{A})\to\mathcal{H}$ as

$$\mathcal{A}(t)U=\begin{pmatrix} v \\ \frac{d}{a}u_{xx}-\frac{\beta}{a}\theta_x \\ \frac{\kappa}{b}\theta_{xx}-\frac{\beta}{b}v_x \\ -\frac{1-\tau'(t)\rho}{\tau(t)}z_\rho(\rho,t) \end{pmatrix},$$

the domain of operator $\mathcal{A}(t)$ is defined by

$$\begin{aligned} D(\mathcal{A}(t))=\{(u,v,\theta,z)^T\in V_3\times V_1\times(H^2(0,L)\cap V_2)\times L^2(\mathbb{R}_+;H^1(0,1))| \\ \theta(L,t)=z(0,t),\quad \theta_x(l,t)+k_1(t)\theta(L,t)+k_2(t)z(1,t)=0\}. \end{aligned} \tag{2.26}$$

Remark 2.3. The operator $\mathcal{A}(t)$ is non-autonomous, and its domain is independent on time t, i.e., $D(\mathcal{A}(t))=D(\mathcal{A}(0))$ for all $t>0$.

Lemma 2.1. *The energy is defined as*

$$E(t)=\frac{1}{2}\int_0^L(au_t^2+du_x^2+b\theta^2)\mathrm{d}x+\frac{\xi(t)\tau(t)}{2}\int_0^1 z^2(\rho,t)\mathrm{d}\rho-\int_\Omega(h_1u+h_2\theta)\mathrm{d}x, \tag{2.27}$$

where

$$\xi(t)=\bar{\xi}k_1(t) \tag{2.28}$$

is a non-increasing function in $C^1(\mathbb{R}_+)$ and

$$\frac{\kappa\alpha}{1-l}<\bar{\xi}<2\kappa\Big(1-\frac{a\alpha}{2}\Big). \tag{2.29}$$

Let (u,θ,z) be a solution of problem (2.19)*. Then the energy functional* (2.27) *satisfies*

$$\begin{aligned} E'(t)\le-\kappa\int_0^L\theta_x^2\mathrm{d}x-\frac{\xi(t)\tau'(t)}{2}\int_0^1 z^2\mathrm{d}\rho-\kappa\Big[k_1(t)-\frac{k_2(t)}{2}-\frac{\xi(t)}{2\kappa}\Big]\theta^2(L) \\ -\frac{1}{2}\Big[\xi(t)(1-\tau'(t)-\kappa k_2(t))\Big]z^2(1,t)\ \le 0. \end{aligned} \tag{2.30}$$

Proof. Multiplying the first and second equations of (2.19) by $u_t(x,t)$ and $\theta(x,t)$ respectively, and integrating by parts on $[0,L]$, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L (au_t^2+du_x^2)\mathrm{d}x=-\int_0^L \beta\theta_x u_t \mathrm{d}x, \tag{2.31}$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L \theta^2\mathrm{d}x+\kappa\int_0^L \theta_x^2\mathrm{d}x+\kappa k_1(t)\theta^2(L)+\kappa k_2(t)\theta(l)z(\rho,L)=\int_0^L \beta\theta_x u_t\mathrm{d}x. \tag{2.32}$$

Multiply the third equation of (2.19) by $\xi(t)z(\rho,t)$ and integrate by parts on [0,1], this results in

$$\tau(t)\xi(t)\int_0^1 z_t(\rho,t)\mathrm{d}\rho=-\frac{\xi(t)}{2}\int_0^1(1-\tau'(t)\rho)\frac{\partial}{\partial\rho}z^2(\rho,t)\mathrm{d}\rho, \tag{2.33}$$

which leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\xi(t)\tau(t)}{2}\int_0^1 z^2(\rho,t)\mathrm{d}\rho\Big)=\frac{\xi'(t)\tau(t)}{2}\int_0^1 z^2(\rho,t)\mathrm{d}\rho+\frac{\xi(t)\tau(t)}{2}\int_0^1\frac{\mathrm{d}}{\mathrm{d}t}z^2(\rho,t)\mathrm{d}\rho \tag{2.34}$$

with

$$\begin{aligned}&\frac{\xi(t)\tau(t)}{2}\int_0^1\frac{\mathrm{d}}{\mathrm{d}t}z^2(\rho,t)\mathrm{d}\rho\\ =&-\frac{\xi(t)}{2}(1-\tau'(t))z^2(1,t)+\frac{\xi(t)}{2}z^2(0,t)-\frac{\xi(t)}{2}\tau'(t)\int_0^1 z^2(\rho,t)\mathrm{d}\rho.\end{aligned} \tag{2.35}$$

The above estimates imply that

$$\begin{aligned}E'(t)=&-\kappa\int_0^L\theta_x^2\mathrm{d}x-\kappa k_1(t)\theta^2(L)-\kappa k_2(t)\theta(l)z(1,t)-\frac{\xi(t)}{2}(1-\tau'(t))z^2(1,t)\\ &-\frac{\xi(t)}{2}\theta^2(L)-\frac{\xi(t)\tau'(t)}{2}\int_0^1 z^2\mathrm{d}\rho\le 0\end{aligned} \tag{2.36}$$

provided that

$$\Big(1-\frac{a\alpha}{2}\Big)k_1(t)-\frac{\xi(t)}{2\kappa}\ge 0, \tag{2.37}$$

$$\xi(t)(1-\tau'(t))-\kappa\alpha k_1(t)\ge 0, \tag{2.38}$$

which are true from (2.29). The proof is complete. □

Lemma 2.2 (Corollary of Hahn-Banach Theorem-See [30])**.** Let X be a Banach space, $F\subset X$ and $\overline{F}\neq X$, then there exists a functional $\phi\in X'$, $\phi\neq 0$ such that

$$<\phi,x>=0,\qquad \forall\, x\in F. \tag{2.39}$$

Remark 2.4. Lemma 2.2 can be used to prove the subset F is dense in X, which only need to consider a linear bounded functional ϕ satisfying $\phi=0$ in F, and then prove $\phi=0$ in X.

Using the semigroup theory and Theorem 2.1, we can obtain the global well-posedness as following theorem.

Theorem 2.3. *Assume that the initial data $U_0\in\mathcal{H}$, and $h_i(\cdot)\in L^2(0,L)$ for $i=1,2$. Then the problem* (1.1) *possesses a unique global solution satisfying*

$$U\in C([0,+\infty);\mathcal{H}) \tag{2.40}$$

under the hypotheses (H1)-(H5).

Furthermore, if $U_0\in D(\mathcal{A}(0))$, then we have the regular solution

$$U\in C([0,+\infty);D(\mathcal{A}(0)))\cap C^1([0,+\infty);\mathcal{H}). \tag{2.41}$$

Proof. Since the operator $\mathcal{A}(t)$ is non-autonomous, we can not use the framework of semigroup to prove our result only. Combining perturbation theory of non-autonomous linear operators by Kato with semigroup technique, we can derive the global well-posedness by verifying Theorem 2.1 as following steps.

Step 1: $D(\mathcal{A}(0))$ **is dense in** $\mathcal{H}$ **and** $D(\mathcal{A}(t))=D(\mathcal{A}(0))$**.**

Lemma 2.3. *The domain $D(\mathcal{A}(0))$ is dense in phase space $\mathcal{H}$.*

Proof. Let $\hat{U}=(\hat{u},\hat{v},\hat{\theta},\hat{z})^T\in\mathcal{H}$ be orthogonal to all elements of $D(\mathcal{A}(0))$, which means

$$(U,\hat{U})_{\mathcal{H}}=\int_0^L(du\hat{u}+a\hat{v}v\mathrm{d}x+b\theta\hat{\theta})\mathrm{d}x+\xi(t)\tau(t)\int_0^1 z\hat{z}\mathrm{d}\rho=0 \tag{2.42}$$

for $U=(u,v,\theta,z)^T\in D(\mathcal{A}(0))$. Taking $u=v=\theta=0$ and $z\in C_0^\infty(0,1)$, as the following result from (2.42) for $U=(0,0,0,z)^T\in D(\mathcal{A}(0))$, therefore

$$\xi(t)\tau(t)\int_0^1\hat{z}z\mathrm{d}\rho=0.$$

Since $C_0^\infty(0,1)$ is dense in $L^2(0,1)$ and $\xi(t)\tau(t)\geq 0$ but not always zero, then it follows $\hat{z}=0$ for all $(\rho,t)\in(0,1)\times\mathbb{R}^+$.

Similarly, taking $u=v=z=0$ for $U=(0,0,\theta,0)^T\in D(\mathcal{A}(0))$, (2.42) and $\theta\in C_0^\infty(0,L)$ lead to

$$\int_0^L b\theta\hat{\theta}\mathrm{d}x=0,$$

which results in $\hat{\theta}=0$ because of $C_0^\infty(0,L)$ is dense in $L^2(0,L)$. Analogously, $\hat{u}=0$ and $\hat{v}=0$ can be obtained. Thus, by using Lemma 2.2, $D(\mathcal{A}(0))$ is dense in $\mathcal{H}$.

Noting that the operator $\mathcal{A}(t)$ independs on time t, this yields $D(\mathcal{A}(t))=D(\mathcal{A}(0))$. □

Step 2: The C_0-semigroup in $\mathcal{H}$ is generated by perturbed operator $\widetilde{\mathcal{A}}(t)$ for $\mathcal{A}(t)$.

Lemma 2.4. *For an arbitrary fixed time t, the perturbed operator $\widetilde{\mathcal{A}}(t)$ for $\mathcal{A}(t)$ generates a C_0-semigroup in $\mathcal{H}$.*

Proof. By using the Lumper-Phillips Theorem, we only need to prove the operator $\widetilde{\mathcal{A}}(t)$ satisfies dissipative and the maximal monotone properties, which is divided into the following procedures.

(1) $\tilde{\mathcal{A}}(t)$ is dissipative.
The time dependent inner produce of $\mathcal{H}$ is defined by

$$\left\langle U,\tilde{U}\right\rangle_t=\int_0^L (du\hat{u}+a\hat{v}v\mathrm{d}x+b\theta\hat{\theta})\mathrm{d}x+\xi(t)\tau(t)\int_0^1 z\hat{z}\mathrm{d}\rho \tag{2.43}$$

for $U=(u,v,\theta,z)^T$ and $\tilde{U}=(\tilde{u},\tilde{v},\tilde{\theta},\tilde{z})^T$. Hence for

$$\mathcal{A}(t)U=\left(v,\frac{d}{a}u_{xx}-\frac{\beta}{a}\theta_x,\frac{\kappa}{b}\theta_{xx}-\frac{\beta}{b}v_x,-\frac{1-\tau'(t)\rho}{\tau(t)}z_\rho\right)^T\in D(\mathcal{A}(t))$$

and fixed time $t>0$, the inner product can be written and estimated as

$$\begin{aligned}\left\langle \mathcal{A}(t)U,U\right\rangle_t=&\int_0^L (dv_xu_x+(du_{xx}-\beta\theta_x)v+(\kappa\theta_{xx}-\beta v_x)\theta)\mathrm{d}x\\&-\xi(t)\tau(t)\int_0^1\frac{1-\tau'(t)\rho}{\tau(t)}z_\rho z\mathrm{d}\rho\\ \le&-\kappa\int_0^L\theta_x^2\mathrm{d}x-\frac{\xi(t)\tau'(t)}{2}\int_0^1 z^2\mathrm{d}\rho-\left(\kappa k_1(t)-\frac{\xi(t)}{2}\right)z^2(0,t)\\&-\frac{\xi(t)(1-\tau'(t))}{2}z^2(1,t)-\kappa k_2(t)z(0,t)z(1,t)\end{aligned} \tag{2.44}$$

with

$$-\kappa k_2(t)z(0,t)z(1,t)\le k_1(t)\frac{\kappa\alpha}{2\sqrt{1-d}}z^2(0,t)+k_1(t)\frac{\kappa\alpha\sqrt{1-d}}{2}z^2(1,t),$$

which results in

$$\begin{aligned}\left\langle \mathcal{A}(t)U,U\right\rangle_t\le&-\kappa\int_0^L\theta_x^2\mathrm{d}x-\left(\kappa k_1(t)-\frac{\xi(t)}{2}-k_1(t)\frac{\kappa\alpha}{2\sqrt{1-d}}\right)z^2(0,t)\\&-\left[\frac{\xi(t)(1-\tau'(t))}{2}-\frac{\xi(t)(1-\tau'(t))}{2}z^2(1,t)-\frac{\xi(t)\tau'(t)}{2}\right]z^2(1,t)\\&+\frac{\xi(t)|\tau'(t)|}{2\tau(t)}\tau(t)\int_0^1 z^2(t,\rho)\mathrm{d}\rho.\end{aligned} \tag{2.45}$$

The estimate (2.45) implies that

$$\left\langle \tilde{\mathcal{A}}(t)U,U \right\rangle_t \leq 0 \tag{2.46}$$

for $\tilde{\mathcal{A}}(t)=\mathcal{A}(t)-\tilde{\kappa}(t)I$ with $\frac{|\tau'(t)|}{2\tau(t)} \leq \tilde{\kappa}(t)=\frac{\sqrt{1+(\tau'(t))^2}}{2\tau(t)}$, this means $\tilde{\mathcal{A}}(t)$ is dissipative.

(2) The maximal monotone property of $\tilde{\mathcal{A}}(t)$ and $\mathcal{A}(t)$.
The purpose is equivalent to prove the surjectivity of operator $\lambda I-\mathcal{A}(t)$ for fixed $t>0$ and some $\lambda>0$ from the Lumper-Phillips Theorem.

Let $F=(f_1,f_2,f_3,f_4)^T \in \mathcal{H}$, by using the Lumper-Phillips Theorem, we only need to seek a solution $U=(u,v,\theta,z)^T \in D(\mathcal{A}(t))$, such that

$$(\lambda I-\mathcal{A}(t))U=F, \tag{2.47}$$

that is, U satisfies the following equations

$$\begin{cases} \lambda u-v=f_1, \\ \lambda a v-du_{xx}+\beta\theta_x=af_2, \\ \lambda b\theta-\kappa\theta_{xx}+\beta v_x=bf_3, \\ \lambda z+\dfrac{1-\rho\tau'(t)}{\tau(t)}z_\rho=f_4. \end{cases} \tag{2.48}$$

By virtue of the similar technique as in [8], the solution of fourth equation for (2.48) with initial data $z(t,0)=\theta(L,t)=\theta(L)$ can be written as

$$z(\rho,t)=\theta(L)e^{\sigma(\rho,t)}+\tau(t)e^{\sigma(\rho,t)}\int_0^\rho \frac{f_4(x,s)}{1-s\tau'(s)}e^{-\sigma(s,t)}\mathrm{d}s \qquad \text{for a fixed } t \text{ and } \tau'(t)\neq 0. \tag{2.49}$$

where

$$\sigma(\rho,t)=\lambda\frac{\tau(t)}{\tau'(t)}\ln(1-\rho\tau'(t)).$$

Otherwise, if $\tau'(t)=0$,

$$z(\rho,t)=\theta(L)e^{-\lambda\tau(t)\rho}+\tau(t)e^{-\lambda\tau(t)\rho}\int_0^\rho f_4(x,s)e^{-\lambda\tau(t)s}\mathrm{d}s \qquad \text{for a fixed } t. \tag{2.50}$$

Choose $\rho=1$, one can derive

$$\begin{aligned} z(1,t)&=\theta(L)e^{\sigma(1,t)}+\tau(t)e^{\sigma(1,t)}\int_0^1 \frac{f_4}{1-s\tau'(t)}e^{-\sigma(s,t)}\mathrm{d}s=\theta(L)e^{\sigma(1,t)}+z_0, \\ z(1,t)&=\theta(L)e^{-\lambda\tau(t)}+\tau(t)e^{-\lambda\tau(t)}\int_0^1 f_4e^{-\lambda\tau(t)s}\mathrm{d}s=\theta(L)e^{-\lambda\tau(t)}+z_1 \end{aligned} \tag{2.51}$$

for $\tau'(t)\neq 0$ and $\tau'(t)=0$ respectively, where

$$z_0=\tau(t)e^{\sigma(1,t)}\int_0^1 \frac{f_4}{1-s\tau'(t)}e^{-\sigma(s,t)}\mathrm{d}s,\quad z_1=\tau(t)e^{-\lambda\tau(t)}\int_0^1 f_4 e^{-\lambda\tau(t)s}\mathrm{d}s.$$

Suppose that u and v have some appropriate regularity, it yields that

$$v=\lambda u-f_1 \tag{2.52}$$

and $v\in H^1(0,L)$ from the first equation in (2.48). Then, the first three equations of (2.48) can be rewritten as

$$\begin{cases}-du_{xx}+\lambda^2 au+\beta\theta_x=f,\\ -\kappa\theta_{xx}+\lambda b\theta+\lambda\beta u_x=\tilde{f}\end{cases} \tag{2.53}$$

with $f=af_2+\lambda af_1$, $\tilde{f}=bf_3+\beta(f_1)_x\in L^2(0,L)$.

The bilinear form of (2.53) for $(u,\theta),(\phi,w)$ in $V_1\times V_2$ is defined by

$$\begin{aligned}\mathcal{B}((u,\theta),(\phi,w))=&\int_0^L\Big[du_x\phi_x+\lambda^2 au\phi+\beta\theta_x\phi+\kappa\theta_x w_x+\lambda b\theta w+\lambda\beta u_x w\Big]\mathrm{d}x\\ &+\kappa k_1(t)w(L)\theta(L)+\kappa k_2(t)w(L)\theta(L)e^{\sigma(1,t)}\end{aligned} \tag{2.54}$$

and

$$\tilde{\mathcal{B}}((u,\theta),(\phi,w))=\int_0^L\Big[du_x\phi_x+\lambda^2 au\phi+\beta\theta_x\phi+\kappa\theta_x w_x+\lambda b\theta w+\lambda\beta u_x w\Big]\mathrm{d}x \tag{2.55}$$

$$+\kappa k_1(t)w(L)\theta(L)+\kappa k_2(t)w(L)\theta(L)e^{-\lambda\tau(t)} \tag{2.56}$$

for $\tau'(t)\neq 0$ and $\tau'(t)=0$ respectively.

Then, consider the variational problem

$$\mathcal{B}((u,\theta),(\phi,w))=\mathcal{F}((\phi,w)), \tag{2.57}$$

$$\tilde{\mathcal{B}}((u,\theta),(\phi,w))=\tilde{\mathcal{F}}((\phi,w)) \tag{2.58}$$

with linear functionals form

$$\mathcal{F}((\phi,w))=\int_0^L\Big(f\phi+\tilde{f}w\Big)\mathrm{d}x+\kappa k_2(t)w(L)\tau(t)e^{\sigma(1,t)}\int_0^1\frac{f_4}{1-s\tau'(t)}e^{-\sigma(s,t)}\mathrm{d}s,\qquad \tau'(t)\neq 0$$

and

$$\tilde{\mathcal{F}}((\phi,w))=\int_0^L\Big(f\phi+\tilde{f}w\Big)\mathrm{d}x+\kappa k_2(t)w(L)\tau(t)e^{-\lambda\tau(t)}\int_0^1 f_4 e^{-\lambda\tau(t)s}\mathrm{d}s,\qquad \tau'(t)=0.$$

Clearly, $\mathcal{B}(w_1,w_2)$ is continuous and coercive in $V_1\times V_2$ with the equipped norm

$$\|(u,\theta)\|^2=\|u\|^2+\|u_x\|^2+\|\theta\|^2+\|\theta_x\|^2. \tag{2.59}$$

Indeed, since assumption $(\mathbf{H_1})$ guarantees that $\sigma(1,t)=\frac{\tau(t)}{\tau'(t)}\ln(1-\tau'(t))<0$ holds, there exists some $\lambda=\lambda_0>0$ satisfies $\frac{(\lambda_0-1)^2\beta^2}{2d}\leq\frac{\lambda_0 b}{2}$, such that the integration by parts results in the coercive property

$$\begin{aligned}\mathcal{B}((u,\theta),(u,\theta))=&\int_0^L\Big[du_x^2+\lambda^2au^2+\beta\theta_xu+\kappa\theta_x^2+\lambda b\theta^2+\lambda\beta u_x\theta\Big]\mathrm{d}x\\&+\kappa k_1(t)\theta^2(L)+\kappa k_2(t)\theta^2(L)e^{\sigma(1,t)}\\\geq&\int_0^L\Big[\frac{d}{2}u_x^2+\lambda_0^2au^2+\kappa\theta_x^2+\frac{\lambda_0 b}{2}\theta^2\Big]\mathrm{d}x+\kappa k_1(t)(1-\alpha)\theta^2(L)\\\geq& C_1\|(u,\theta)\|^2\end{aligned}\tag{2.60}$$

and

$$\begin{aligned}\tilde{\mathcal{B}}((u,\theta),(u,\theta))=&\int_0^L\Big[du_x^2+\lambda^2au^2+\beta\theta_xu+\kappa\theta_x^2+\lambda b\theta^2+\lambda\beta u_x\theta\Big]\mathrm{d}x\\&+\kappa k_1(t)\theta^2(L)+\kappa k_2(t)\theta^2(L)e^{-\lambda\tau(t)}\\\geq& C_1\|(u,\theta)\|^2.\end{aligned}\tag{2.61}$$

Therefore, by Lax-Milgram theorem, the problem (2.53) possesses a solution for $(u,\theta)\in V_1\times V_2$, i.e., for any (ϕ,w) in the space $C_0^\infty(0,L)\times C_0^\infty(0,L)$ which is dense in $V_1\times V_2$, the variational forms (2.57) and (2.58) hold. Since $\mathcal{F}((\phi,w)),\tilde{\mathcal{F}}((\phi,w))\in L^2(0,L)$, then we can show that (u,θ) satisfy the boundary conditions and $v=\lambda u-f_1\in V_1$. Consequently, from the L^2-theory of elliptic equations, the abstract problem (2.47) possesses a unique solution $U\in D(\mathcal{A}(t))$ for a fixed $t>0$. Since the boundedness of $\tilde{\kappa}(t)>0$, we can conclude that $\mathcal{A}(t)-\tilde{\kappa}(t)I$ is surjective, which leads to $\tilde{\mathcal{A}}(t)$ is maximal for a fixed $t>0$. Then the proof is complete. □

Step 3: The stable property of $\tilde{\mathcal{A}}(t)$.

Lemma 2.5. *The family of non-autonomous operators $\widetilde{\mathcal{A}}(t)$ is stable in the phase space $\mathcal{H}$ for $t\in[0,T]$.*

Proof. For $U=(u,v,\theta,z)^T\in\mathcal{H}$, by the theory of Kato as Theorem 2.1, we only need to verify that

$$\frac{\|U\|_t}{\|U\|_s}\leq e^{\frac{l}{\tau_0}|t-s|}\qquad\text{for all } t,s\in[0,T].\tag{2.62}$$

Let $\|U\|_t^2=\|(u,v,\theta,z)\|_{\mathcal{H}}^2$ be the norm associated with inner product (2.24). Then, it yields

$$\begin{aligned}&\|U\|_t^2-\|U\|_s^2 e^{\frac{l}{\tau_0}|t-s|}\\=&(1-e^{\frac{\widetilde{\tau}_1}{\tau_0}(t-s)})\Big[\|u_x\|_2^2+\|u_t\|_2^2+\|\theta\|^2\Big]+\Big(\xi(t)\tau(t)-\xi(s)\tau(s)e^{\frac{l}{\tau_0}|t-s|}\Big)\int_0^1 z^2(t)\mathrm{d}\rho\end{aligned}$$

for $s,\ t\in[0,T]$ and $0\le s\le t\le T$. The non-negative parameter $\frac{l}{\tau_0}|t-s|$ implies that $1-e^{\frac{l}{\tau_0}(t-s)}\le 0$. Hence, we only need to prove

$$\xi(t)\tau(t)-\xi(s)\tau(s)e^{\frac{l}{\tau_0}|t-s|}\le 0. \tag{2.63}$$

Since $\tau(t)\in W^{2,\infty}([0,T])\hookrightarrow C^1([0,T])$, it is easy to see that $\rho(t)$ can be represented by

$$\tau(t)=\tau(s)+\tau'(r)(t-s),\qquad r\in(s,t).$$

Then by the monotone property $\xi'\le 0$ and $\xi>0$, we have

$$\xi(t)\tau(t)\le\xi(s)\tau(s)+\xi(s)\tau'(r)(t-s),$$

which results in

$$\frac{\xi(t)\tau(t)}{\xi(s)\tau(s)}\le 1+\frac{|\tau'(r)|}{\tau(s)}|t-s|\le 1+\frac{l}{\tau_0}|t-s|\le e^{\frac{l}{\tau_0}|t-s|}$$

from (2.15) and (2.16). This shows that (2.62) is true. In addition, the trivial case for $\|U\|_\cdot\equiv 0$ holds obviously.

Hence, combining Lemma 2.4 and (2.62), we can conclude that $\widetilde{\mathcal{A}}(t)$ is stable in $\mathcal{H}$ by using Proposition 1.1 in [25]. □

Step 4: The uniform bounded of $\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{\mathcal{A}}(t)$.

Lemma 2.6. *The uniform bounded $\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{\mathcal{A}}(t)\in L_*^\infty([0,T],B(D(\mathcal{A}(0)),\mathcal{H}))$ holds, here*

$$L_*^\infty([0,T],B(D(\mathcal{A}(0)),\mathcal{H}))$$

is the space of equivalent class of essentially boundedness, measurable functions from $[0,T]$ to $B(D(\mathcal{A}(0)),\mathcal{H})$, $B(D(\mathcal{A}(0)),\mathcal{H})$ is the set of all bounded linear functionals from $D(\mathcal{A}(0))$ to $\mathcal{H}$.

Proof. Noting that

$$\mathcal{A}(t)=\begin{pmatrix}0&1&0&0\\ \dfrac{d}{a}\partial_{xx}&0&-\dfrac{\beta}{a}\partial_x&0\\ 0&-\dfrac{\beta}{b}\partial_x&\dfrac{\kappa}{b}\partial_{xx}&0\\ 0&0&0&-\dfrac{1-\rho\tau'(t)}{\tau(t)}\dfrac{\partial}{\partial\rho}\end{pmatrix},$$

we deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{A}(t)U=\begin{pmatrix} 0\\ 0\\ 0\\ \dfrac{\rho\tau(t)\tau''(t)-\tau'(t)(\rho\tau'(t)-1)}{\tau^2(t)}z_\rho \end{pmatrix} \tag{2.64}$$

is also bounded for $t\in\mathbb{R}^+$ because $\frac{\rho\tau(t)\tau''(t)-\tau'(t)(\rho\tau'(t)-1)}{\tau^2(t)}$ is bounded on $[0,T]$ from (2.15)-(2.16). Hence, (2.64) results in the boundedness of

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{\mathcal{A}}(t)U=\Big(\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{A}(t)-\tilde{\kappa}'(t)\Big)U$$

for $U\in D(\mathcal{A}(0))$ and $t\in\mathbb{R}^+$ because

$$\tilde{\kappa}'(t)=\frac{\tau'(t)\tau''(t)}{2\tau(t)\sqrt{1+(\tau'(t))^2}}+\frac{\tau'(t)\sqrt{1+(\tau'(t))^2}}{2\tau^2(t)}$$

is bounded from (2.15)-(2.16), where $\tilde{\kappa}(t)=\dfrac{\sqrt{1+(\tau'(t))^2}}{2\tau(t)}$.

The proof of this lemma is complete. □

Step 5: The local mild solution Some lemmas are needed to verify the local existence of mild solution for our problem.

Lemma 2.7. *Assume the hypotheses* (H1)-(H5) *hold. Let* $\{\Lambda(t)\}=-\{\mathcal{A}(t)\}$ *be a unbounded operators family in* $\mathcal{H}$. *Then,* $\{\Lambda(t)\}$ *is stable, which also Lipschitz in* $D(\mathcal{A}(t))$.

Proof. By the definition of $\tilde{\mathcal{A}}(t)=\mathcal{A}(t)-\kappa(t)I$ with appropriate $\kappa(t)$ such that $\tilde{\mathcal{A}}(t)$ is positive and self-adjoint, we can obtain that $\tilde{\mathcal{A}}(t)$ from Lemma 2.5. Noting that the perturbation of $\mathcal{A}(t)$ is linear, using the technique in [27], we conclude that $\mathcal{A}(t)$ is stable in $\mathcal{H}$.

Since the perturbation is linear, for fixed $U\in D(\mathcal{A}(t))$, we have

$$\|[\mathcal{A}(t)-\mathcal{A}(s)]U\|_{D((\mathcal{A})(t))}\leq\left(\frac{(1-\rho^2)l+\tilde{l}\tau_1}{\tau_0^2}\right)|t-s|\|U\|_{\mathcal{H}}, \tag{2.65}$$

which implies the Lipschitz property. □

Lemma 2.8. *Assume the hypothesis* (H4) *holds. Then, the non-autonomous nonlinear term* $H(u,\theta)$ *are Lipschitz in* $\mathcal{H}$. *Moreover, the term* $H(u,\theta)$ *is also uniform bounded in* $\mathcal{H}$.

Proof. From (2.12) in assumption (H3), by virtue of the similar technique of Propositions 2.7 and 2.8 in [27], the Lipschitz continuity and uniform boundedness of $H(u,\theta)$ in $\mathcal{H}$ can be derived easily. □

Lemma 2.9. *Assume that the initial data $U_0 \in \mathcal{H}$, and $h_i(\cdot) \in L^2(0,L)$ for $i=1,2$. Then the problem*

$$\begin{cases} \dfrac{\mathrm{d}U}{\mathrm{d}t} = \mathcal{A}(t)U + H(u,\theta), \\ U(0) = U_0, \end{cases} \tag{2.66}$$

possesses a local solution in $\mathcal{H}$ under the hypotheses (H1)-(H5).

Proof. Combining the Lemmas 2.7-2.8, using the Theorems 2.1 and 2.2, we can conclude the local existence of mild solution. □

Step 6: The global mild solution

(1) The energy of problem should be redefined as (2.27) which is important to extend local solutions. Using the similar technique as in Lemma 2.1, the energy $E(t)$ is non-increasing along any solution $U(t)$ which satisfies that there exist $C_0 > 0$, such that

$$\|U(t)\|^2_{\mathcal{H}} \le C_{0,h_1,h_2}(E(t)+1), \qquad \text{for all } t \ge 0. \tag{2.67}$$

(2) From Lemmas 2.3-2.6 and Kato's theory of operator perturbation, the abstract Cauchy problem

$$\begin{cases} \dfrac{\mathrm{d}\widetilde{U}}{\mathrm{d}t} = \widetilde{\mathcal{A}}(t)\widetilde{U}, \\ \widetilde{U}(0) = U_0 \end{cases} \tag{2.68}$$

has a unique solution

$$\widetilde{U}(t) = e^{\int_0^t \widetilde{\mathcal{A}}(s)\mathrm{d}s} U_0 \in C([0,T], D(\mathcal{A}(0))) \cap C^1([0,T],\mathcal{H}).$$

Hence, $U(t) = e^{\int_0^t \kappa(s)\mathrm{d}s}\widetilde{U}(t)$ is the solution of problem

$$\begin{cases} \dfrac{\mathrm{d}U}{\mathrm{d}t} = \mathcal{A}(t)U, \\ U(0) = U_0, \end{cases} \tag{2.69}$$

because of

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}U(t) &= \kappa(t) e^{\int_0^t \kappa(s)\mathrm{d}s}\widetilde{U}(t) + e^{\int_0^t \kappa(s)\mathrm{d}s}\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{U}(t) \\ &= e^{\int_0^t \kappa(s)\mathrm{d}s}(\kappa(t)I + \widetilde{\mathcal{A}}(t))\widetilde{U}(t) = \mathcal{A}(t)U(t). \end{aligned}$$

Consider the non-homogeneous problem (2.66). Then, the unique mild solution of (2.66) can be computed as

$$U(t) = e^{\int_0^t \mathcal{A}(s)\mathrm{d}s}U_0 + \int_0^t e^{\int_0^{t-s}\mathcal{A}(\sigma)\mathrm{d}\sigma} H(u,\theta)\mathrm{d}s, \qquad \text{for } t \in (0, t_{\max}) \tag{2.70}$$

from (2.69), the estimate (2.67) and Lemmas 2.7-2.9 guarantee that the local solution can be extended to global solution.

The proof is completed. □

3 Uniform stability and dynamics

3.1 The theory of quasi-stability and finite-dimensional attractors

In this section, the theory of finite dimensional global and exponential attractors will be presented for preparation, which are originated from Chueshov and Lasiecka [22].

Definition 3.1 (Chueshov and Lasiecka [22])**.** A dynamics system $(X,S(t))$ is called gradient system if there exists a Lyapunov functional Φ satisfies
(1) $\Phi(S(t)U)$ is non-increasing with respect to $t\geq 0$ for any $U\in X$.
(2) The stationary points of Φ are fixed points of $S(t)$, i.e., $\Phi(S(t)z)=\Phi(z)$ for all t.

Definition 3.2 (Chueshov and Lasiecka [22])**.** The dynamic system $(X,S(t))$ is called quasi-stable on a set $B\subset X$ if there exists a compact semi-norm n_X on X, and nonnegative scalar functions $a(t)$, $c(t)$ are locally bounded on $[0,\infty)$, $b(t)\in L^1(\mathbb{R}^+)$ and $\lim_{t\to\infty}b(t)=0$, such that

$$\|S(t)U_1-S(t)U_2\|_X^2\leq a(t)\|U_1-U_2\|_X^2 \tag{3.1}$$

and

$$\|S(t)U_1-S(t)U_2\|_X^2\leq b(t)\|U_1-U_2\|_X^2+c(t)\sup_{0<s<t}[n_X(u_1(s)-u_2(s))]^2 \tag{3.2}$$

for U_1, $U_2\in B$.

Theorem 3.1 (Chueshov and Lasiecka [22])**.** *Let $(X,S(t))$ be a gradient system and suppose that the system is quasi-stable on every bounded positively invariant set $B\subset X$. Then $(X,S(t))$ has a global attractor $\mathcal{A}=M_+(\mathcal{N})$ with finite fractal dimension, here $\mathcal{N}$ is the set of equilibrium for $S(t)$, $M_+(\mathcal{N})$ is the unstable manifold for $\mathcal{N}$. Moreover, the generalized finite fractal dimensional exponential attractor also exists under suitable condition for $S(t)$.*

3.2 Gradient system

Based on the global well-posedness in Theorem 2.3, under the hypotheses (H1)-(H5), the solution for problem (2.19) generates a dynamic system $(\mathcal{H},S(t))$, where $S(t):\mathcal{H}\to\mathcal{H}$ is a solution semigroup. In this section, we will prove that $(\mathcal{H},S(t))$ is a gradient system, which possesses a Lyapunov functional.

Lemma 3.1. *For $U_0=(u_0,u_1,\theta_0,f_0)^T\in\mathcal{H}$, the dynamic system $(\mathcal{H},S(t))$ is a gradient system.*

Proof. The existence of Lyapunov functional is $E(t)$. Denote $\Phi(S(t)U)=E(t)$ along the solution trajectory $S(t)U=(u(t),v(t),\theta(t),z(t))^T$, which is non-increasing with respect to t from Lemma 2.1, assume

$$\Phi(S(t)U_0)=S(t)U_0$$

for all $t\geq 0$ and $U_0=(u_0,u_1,\eta_0,f_0)^T\in\mathcal{H}$. Then $E'(t)=0$ implies

$$-\kappa\int_0^L\theta_x^2\mathrm{d}x-\frac{\xi(t)\tau'(t)}{2}\int_0^1 z^2\mathrm{d}\rho-\kappa\Big[k_1(t)-\frac{k_2(t)}{2}-\frac{\xi(t)}{2\kappa}\Big]\theta^2(L)$$
$$-\frac{1}{2}\Big[\xi(t)(1-\tau'(t)-\kappa k_2(t))\Big]z^2(1,t)=0. \tag{3.3}$$

Since every term on the LHS of (3.3) is non-positive, we can conclude that

$$U_0=S(t)(0,\theta_0,0,0)^T=(0,\theta_0,0,0)^T$$

is the fixed point. Then, $(\mathcal{H},S(t))$ is a gradient system. □

3.3 Asymptotic stability and quasi-stability

In this section, the quasi-stability is given by multiplier approach, then the uniform stability can be obtained by the similar technique.

• Quasi-stability

Theorem 3.1. *The dynamic system $(\mathcal{H},S(t))$ has the property of quasi-stability on a set $B\subset\mathcal{H}$ as in Definition* 3.2 *under the hypotheses* (H1)-(H5)*.*

Proof. Suppose that $U_i=(u_i,v_i,\theta_i,z_i)^T(i=1,2)$ are two solutions of the problem (2.19) with initial data $U_{i0}=(u_{i0},u_{i1},\theta_{i0},f_{i0})^T$, let $\bar{U}=U_1-U_2=(\bar{u},\bar{v},\bar{\theta},\bar{z})$ be the difference with initial data $U_0=(\bar{u}_0,\bar{u}_1,\bar{\theta}_0,\bar{f}_0)^T$. Then it is easy to check that $\bar{U}$ satisfies

$$\begin{cases}a\bar{u}_{tt}-d\bar{u}_{xx}+\beta\bar{\theta}_x+(h_1(u_1)-h_1(u_2))=0,\\ b\bar{\theta}_t-\kappa\bar{\theta}_{xx}+\beta\bar{u}_{xt}+(h_2(\theta_1)-h_2(\theta_2))=0,\\ \tau(t)\bar{z}_t(\rho,t)+(1-\rho\tau'(t))\bar{z}_\rho(\rho,t)=0, & \text{in } (0,1)\times\mathbb{R}^+,\\ \bar{u}_x(0,t)=\bar{u}(L,t)=\bar{\theta}(0,t)=0, & t\geq 0,\\ \bar{\theta}_x(L,t)+k_1(t)\bar{z}(0,t)+k_2(t)\bar{z}(1,t)=0, & t\geq 0,\\ \bar{u}(x,0)=\bar{u}_0(x),\ \bar{u}_t(x,0)=\bar{u}_1(x),\ \bar{\theta}(x,0)=\bar{\theta}_0(x), & t\geq 0,\\ \bar{\theta}(L,t-\tau(0))=\bar{f}_0(L,t-\tau(0)), & \text{in } (0,L)\times(0,\tau(0)).\end{cases} \tag{3.4}$$

The energy functional for problem (3.4) is defined as

$$G(t)=\frac{1}{2}\int_0^L(a\bar{u}_t^2+d\bar{u}_x^2+b\bar{\theta}^2)\mathrm{d}x+\frac{\xi(t)\tau(t)}{2}\int_0^1\bar{z}^2(\rho,t)\mathrm{d}\rho,$$

which is non-increasing and equivalent to the norm $\|U_1-U_2\|_{\mathcal{H}}^2$ because of (2.28)-(2.29) by using the similar technique as in Lemma 2.1, i.e., there exist C_3, $C_4>0$, such that

$$C_3\|U_1-U_2\|_{\mathcal{H}}^2\leq G(t)\leq C_4\|U_1-U_2\|_{\mathcal{H}}^2.$$

Hence, the inequality (3.1) holds. Next, we only need o to prove (3.2) in Definition 3.2 as the following Lemma 3.2, which is the key point here. □

Lemma 3.2. *Let U_1, U_2 be two solutions of problem* (2.19) *with initial data U_{10}, U_{20} in bounded subset $B\subset\mathcal{H}$. Then there exist positive constants γ, B_0, B_1, depending only on B, such that*

$$\|U_1-U_2\|_{\mathcal{H}}^2\le B_0e^{-\gamma t}\|U_{10}-U_{20}\|_{\mathcal{H}}^2+B_1\sup_{0<s<t}\|u_1(s)-u_2(s)\|_4^2. \tag{3.5}$$

Proof. The proof can be transformed into the estimate of $G(t)$ because $G(t)$ and $\|U_1-U_2\|_{\mathcal{H}}^2$ are equivalent, which is divided into the following lemmas by using multiplier method. □

Lemma 3.3. *Define the functional $\phi(t)$ as*

$$\phi(t)=a\int_0^L\bar{u}_t(t)\bar{u}(t)\mathrm{d}x.$$

Then, there exists a constant $m_0>0$, such that

$$\phi'(t)\le-\frac{d}{2}\int_0^L\bar{u}_x^2\mathrm{d}x+\frac{\beta\tilde{\Lambda}}{2d}\int_0^L\bar{\theta}_x^2\mathrm{d}x+a\int_0^L\bar{u}_t^2\mathrm{d}x+\bar{C}_{h_1}\|\bar{u}\|_{L^4}^2. \tag{3.6}$$

Proof. Noting that,

$$\phi'(t)=a\int_\Omega\bar{u}_{tt}\bar{u}\mathrm{d}x+a\int_0^L\bar{u}_t^2\mathrm{d}x$$

and the estimate

$$\begin{aligned}\int_0^L a\bar{u}_{tt}\bar{u}\mathrm{d}x&=\int_0^L(d\bar{u}_{xx}-\beta\bar{\theta}_x+(h_1(u_1)-h_1(u_2)))\bar{u}\mathrm{d}x\\&\le-\frac{d}{2}\int_0^L\bar{u}_x^2\mathrm{d}x+\frac{\beta\tilde{\Lambda}}{2d}\int_0^L\bar{\theta}_x^2\mathrm{d}x+\bar{C}_{h_1}\|\bar{u}\|_{L^4}^2\end{aligned} \tag{3.7}$$

is true from hypothesis (H3) and the global well-posedness, we can derive the desired result. □

Lemma 3.4. *Define the functional $\varphi(t)$ as*

$$\varphi(t)=-ab\int_0^L\bar{\theta}\int_0^x\bar{u}_t(s,t)\mathrm{d}s\mathrm{d}x.$$

Then there exist parameters $m_1>0$ and $m_2>0$, such that

$$\varphi'(t)\le-\frac{m_1}{2}\int_0^L\bar{u}_t^2\mathrm{d}x+2\delta\int_0^L\bar{u}_x^2\mathrm{d}x+m_2\int_0^L\bar{\theta}_x^2\mathrm{d}x+C_\varepsilon[k_1^2(t)\bar{\theta}^2(L,t)+k_2^2(t)\bar{z}^2(1,t)]. \tag{3.8}$$

Proof. The simple computation yields

$$\begin{aligned}\varphi'(t)=&-a\beta\int_0^L \bar{u}_t^2\mathrm{d}x+b\beta\int_0^L\bar{\theta}^2\mathrm{d}x-bd\int_0^L\bar{\theta}\bar{u}_x\mathrm{d}x+a\kappa\int_0^L\bar{\theta}_x\bar{u}_t\mathrm{d}x\\&+a\kappa\Big(k_1(t)\bar{\theta}(L,t)+k_2(t)\bar{z}(1,t)\Big)\Big)\int_0^L\bar{u}_t\mathrm{d}x\\&+a\int_0^L(h_2(\theta_1)-h_2(\theta_2))\mathrm{d}x\int_0^L\bar{u}_t\mathrm{d}x\\&+b\int_0^L\bar{\theta}\int_0^x(h_1(u_1)-h_1(u_2))\mathrm{d}s\mathrm{d}x.\end{aligned}$$

Noting that the estimates

$$a\int_0^L(h_2(\theta_1)-h_2(\theta_2))\mathrm{d}x\int_0^L\bar{u}_t\mathrm{d}x\le\frac{m_1}{2}\|\bar{u}_t\|^2+aL\tilde{C}_1\|\bar{\theta}\|_{L^4}^2, \tag{3.9}$$

$$b\int_0^L\bar{\theta}\int_0^x(h_1(u_1)-h_1(u_2))\mathrm{d}s\mathrm{d}x\le\delta\|\bar{u}_x\|^2+bLC_\delta\tilde{C}_2\|\bar{\theta}\|_{L^4}^2 \tag{3.10}$$

are true from hypothesis (H3), using the embedding $H^1(0,L)\hookrightarrow L^4(0,L)$, we can arrive the desired estimate (3.8) by the Poincaré and Young inequalities. □

To perturb the fourth term in $G(t)$ for using the multiplier approach, the functional $J(t)$ and perturbed energy $\widetilde{G}(t)$ can be defined as

$$\begin{aligned}&J(t)=\bar{\xi}\tau(t)\int_0^1 e^{-\tau(t)\rho}\bar{z}^2(\rho,t)\mathrm{d}\rho,\\&\widetilde{G}(t)=N_1G(t)+\phi(t)+\varepsilon_1\varphi(t)+J(t),\end{aligned}$$

where N_1 is sufficiently large and ε_1 is an appropriate small parameter.

The estimate of $J(t)$ is given in following lemma.

Lemma 3.5. *There exists a parameter $C_J>0$ such that the estimate*

$$J'(t)\le-\bar{\xi}(1-\tau'(t))e^{-\tau(t)}\bar{z}^2(1,t)+\bar{\xi}\bar{\theta}^2(L,t)-C_J\bar{\xi}\int_0^1\bar{z}^2(\rho,t)\mathrm{d}\rho \tag{3.11}$$

holds for $\bar{\xi}>0$.

Proof. The estimate of $J'(t)$ can be proceeded by

$$\begin{aligned}J'(t)=&\bar{\xi}\int_0^1\frac{\mathrm{d}}{\mathrm{d}t}(\tau(t)e^{-\rho\tau(t)})\bar{z}^2(\rho,t)\mathrm{d}\rho+\bar{\xi}\int_0^1\frac{\mathrm{d}}{\mathrm{d}\rho}((1-\rho\tau'(t))e^{-\rho\tau(t)})\bar{z}^2(\rho,t)\mathrm{d}\rho\\&-\bar{\xi}(1-\tau'(t))e^{-\tau(t)}z^2(1,t)+\bar{\xi}\bar{z}^2(0,t).\end{aligned}$$

Since

$$\frac{\mathrm{d}}{\mathrm{d}t}(\tau(t)e^{-\rho\tau(t)})+\frac{\mathrm{d}}{\mathrm{d}\rho}((1-\rho\tau'(t))e^{-\rho\tau(t)})=-2\tau(t)e^{-\rho\tau(t)}<0, \tag{3.12}$$

this lemma holds from choosing appropriate parameter $C_J>0$. □

The equivalence of $G(t)$ and $\widetilde{G}(t)$ shall be illustrated as following.

Lemma 3.6. *For large enough M_1 and M_2, small sufficient ε, there exist $K_1,K_2>0$, such that*

$$K_1G(t)\le\widetilde{G}(t)\le K_2G(t).$$

Proof. By the Hölder and Young inequalities, it is easy to check the equivalence in this lemma. □

• *Proof of Lemma* 3.2*:* From Lemmas 3.3-3.5, one can derive

$$\begin{aligned}\widetilde{G}'(t)=&N_1G'(t)+\phi'(t)+\varepsilon_1\varphi'(t)+J'(t)\\ \le&-\Big(\kappa N_1-\frac{\beta\tilde{\Lambda}}{2d}-m_2\varepsilon_1\Big)\int_0^L\bar{\theta}_x^2\mathrm{d}x-\Big(\frac{\xi(t)\tau'(t)N_1}{2}+C_J\bar{\xi}\Big)\int_0^1\bar{z}^2\mathrm{d}\rho\\ &-\Big\{\frac{N_1}{2}\Big[\xi(t)(1-\tau'(t))-\kappa\alpha k_1(t)\Big]+\bar{\xi}(1-\tau'(t))e^{-\tau(t)}-C_\varepsilon\varepsilon_1\alpha^2k_1^2(t)\Big\}\bar{z}^2(1,t)\\ &-\Big(\frac{d}{2}-\delta\varepsilon_1\Big)\int_0^L\bar{u}_x^2\mathrm{d}x-\Big(m_1\varepsilon_1-a\Big)\int_0^L\bar{u}_t^2\mathrm{d}x\\ &-\Big\{\kappa N_1\Big[k_1(t)-\frac{a\alpha k_1(t)}{2}-\frac{\xi(t)}{2\kappa}\Big]-\bar{\xi}-C_\varepsilon\varepsilon_1k_1^2(t)\Big\}\bar{\theta}^2(L,t)+\mathcal{M}(t),\end{aligned} \tag{3.13}$$

where $\mathcal{M}(t)=\bar{C}_{h_1}\|\bar{u}\|^2_{L^4}$.

Suppose that N_1 is large enough, choose appropriate ε_1 and $\delta>0$ small enough, it is easy to check that

$$\begin{aligned}&\kappa N_1-\frac{\beta\tilde{\Lambda}}{2d}-m_2\varepsilon_1>0,\\ &\frac{N_1}{2}\Big[\xi(t)(1-\tau'(t))-\kappa\alpha k_1(t)\Big]+\bar{\xi}(1-\tau'(t))e^{-\tau(t)}-C_\varepsilon\varepsilon_1\alpha^2k_1^2(t)>0,\\ &\frac{d}{2}-\delta\varepsilon_1>0,\qquad m_1\varepsilon_1-a>0.\end{aligned}$$

For our purpose, we only need

$$\kappa N_1\Big[k_1(t)-\frac{a\alpha k_1(t)}{2}-\frac{\xi(t)}{2\kappa}\Big]-\bar{\xi}-C_\varepsilon\varepsilon_1k_1^2(t)>0,$$

i.e., $(1-\frac{a\alpha}{2})k_1(t)-\frac{\xi(t)}{2\kappa}>0$ is guaranteed by (2.37). Then, all coefficients on RHS of (3.13) except $M(t)$ are negative.

Hence, there exists $K_3>0$, such that

$$\widetilde{G}'(t)\le -K_3G(t)+\mathcal{M}(t),$$

which results in

$$\widetilde{G}'(t)\le -\frac{K_3}{K_1}\widetilde{G}(t)+\mathcal{M}(t) \tag{3.14}$$

for all $t\ge 0$ by Lemma 3.6. Apply Gronwall's lemma to (3.14), it yields

$$\widetilde{G}(t)\le \widetilde{G}(0)e^{-\frac{K_3}{K_1}t}+\int_0^t e^{-\frac{K_3}{K_1}(t-s)}M(s)\mathrm{d}s,$$

which leads to

$$G(t)\le \frac{K_2}{K_1}G(0)e^{-\frac{K_3}{K_1}t}+\frac{K_1\bar{C}_{h_1}}{K_3}\sup_{0<s<t}\|\bar{u}\|_{L^4}^2.$$

By the uniform boundedness of U in $\mathcal{H}$, and the equivalence between $G(t)$ and $\|U_1-U_2\|_{\mathcal{H}}^2$, we conclude that

$$\|U_1-U_2\|_{\mathcal{H}}^2\le \frac{C_4K_2}{K_1}e^{-\frac{K_3}{K_1}t}\|U_{10}-U_{20}\|_{\mathcal{H}}^2+\frac{K_1\bar{C}_{h_1}}{C_3K_3}\sup_{0<s<t}\|u_1(s)-u_2(s)\|_4^2$$

for all $t\ge 0$, which results in the quasi-stability of Lemma 3.2. □

• Uniform asymptotic stability

Theorem 3.2. *Suppose that* $U_0=(u_0,u_1,\theta_0,f_0)^T\in\mathcal{H}$. *Then there exists* $\hat{C}>0$, *such that the uniform asymptotic stability*

$$\|U(t)\|_{\mathcal{H}}^2=\|(u,u_t,\theta,z)\|_{\mathcal{H}}^2\le \hat{C}(1+\|(u_0,u_1,\theta_0,f_0)\|_{\mathcal{H}}^4)$$

for semilinear problem (2.19) *holds under the hypotheses* (H1)-(H5).

Proof. From Lemma 2.1, there exist positive constants $\Upsilon>0$ and $\check{C}_i>0$ for $i=1,2$, we can see that $E'(t)\le 0$ and

$$\Upsilon\|U\|_{\mathcal{H}}^2-\check{C}_1\le E(t)\le \check{C}_2(1+\|U\|_{\mathcal{H}}^4)$$

for $U=(u(t),u_t(t),\theta(t),z(t))^T\in\mathcal{H}$ and $t\ge 0$.

Define the functionals as

$$\begin{aligned}\phi_1(t)&=a\int_0^L u_t(t)u(t)\mathrm{d}x,\\ \varphi_1(t)&=-ab\int_0^L\theta\int_0^x u_t(s,t)\mathrm{d}s\mathrm{d}x,\end{aligned}$$

$$J_1(t)=\bar{\xi}\tau(t)\int_0^1 e^{-\tau(t)\rho}z^2(\rho,t)\mathrm{d}\rho,$$

let

$$\widetilde{E}(t)=N_2E(t)+\phi_1(t)+\varepsilon_2\varphi_1(t)+J_1(t),$$

which is equivalent to $E(t)$ for sufficiently large N_2 and appropriate small $\varepsilon>0$, use the similar multiplier technique as in Theorem 3.1, we can achieve

$$\widetilde{E}'(t)\le -\check{C}_3E(t)+\check{C}_4.$$

Combining the equivalence of $\widetilde{E}(t)$ and $E(t)$, by using Gronwall's lemma, we can conclude that there exists $\check{C}>0$, such that

$$\|(u(t),u_t(t),\theta(t),z(t))\|_{\mathcal{H}}^2\le \check{C}(1+\|(u_0,u_1,\theta_0,f_0)\|_{\mathcal{H}}^4)$$

for $t\ge 0$ and $U_0\in\mathcal{H}$. The proof is complete. □

3.4 Dynamics

In this subsection, the existence of global and exponential attractors can be obtained by virtue of uniform asymptotic stability and the quasi-stability of gradient system, which leads to the asymptotic smoothness.

Theorem 3.3. *Assume that $U_0=(u_0,u_1,\theta_0,f_0)^T\in\mathcal{H}$ and hypothesis* (H1)-(H5) *hold. Then the gradient system $(\mathcal{H},S(t))$ for the problem* (2.19) *possesses a finite fractal dimensional global attractor $\mathcal{A}\subset\mathcal{H}$, which is consisted by the unstable manifold $\mathbb{M}^u(\mathcal{N})$, where $\mathcal{N}$ is the set of stationary points.*

Moreover, the exponential attractor $\mathcal{A}^{exp}\subset\mathcal{H}$ with finite fractal dimension for $(\mathcal{H},S(t))$ is also obtained for our gradient system.

Proof. In order to use Theorems 2.1 to attain the existence of global attractors for problem (2.19), the proof is divided into three steps.

Step 1: A suitable Lyapunov functional has been defined, which results in the dynamical system $(\mathcal{H},S(t))$ is gradient.

Step 2: Note that $\|\cdot\|_4$ is a compact semi-norm in $H_0^1(\Omega)$ because of $H_0^1(\Omega)\hookrightarrow\hookrightarrow L^4(\Omega)$, then the quasi-stability is presented in Theorem 3.1, which guarantees $(\mathcal{H},S(t))$ is asymptotically smooth.

Step 3: Suppose that $U(t)=(u(t),u_t(t),\theta(t),z(t))^T$ is the stationary solution for problem (2.19), which satisfies

$$\begin{cases}-du_{xx}+\beta\theta_x+h_1(u)=0, & \text{in } (0,L)\times(0,\infty),\\ -\kappa\theta_{xx}+h_2(\theta)=0, & \text{in } (0,L)\times(0,\infty).\end{cases}\tag{3.15}$$

Then

$$d\|u_x\|_2^2+\kappa\|\theta_x\|_2^2+\int_0^L h_1(u)u\mathrm{d}x+\int_0^L h_2(\theta)\theta\mathrm{d}x\le -\beta\int_0^L u\theta_x\mathrm{d}x.$$

Since

$$\int_0^L h_1(u)u\mathrm{d}x\ge -\ell_0\tilde{\Lambda}\|u\|^2-\rho_{h_1}L,$$
$$\int_0^L h_2(\theta)\theta\mathrm{d}x\ge -\ell_0\tilde{\Lambda}\|\theta\|^2-\rho_{h_2}L,$$

there exists a positive constant $C'=C'(L,\ell_0,d,\kappa,\tilde{\Lambda},\beta,\rho_{h_i})$, such that

$$d\|u_x\|_2^2+\kappa\|\theta_x\|_2^2\le C',$$

which implies all stationary solutions are uniformly bounded in $\mathcal{H}$.

The estimate (2.67) implies that $\|U\|_{\mathcal{H}}\to\infty$ leads to $\Phi(S(t)U)\to\infty$. Conversely, the estimates in uniform stability implies $\|U\|_{\mathcal{H}}\to\infty$ as $\Phi(S(t)U)\to\infty$.

Hence, the set of stationary solutions for problem (2.19) is bounded in $\mathcal{H}$. In addition, $\Phi(S(t)U)\to\infty$ if and only if $\|U\|_{\mathcal{H}}\to\infty$.

In conclusion, all conditions of Theorem 3.1 have been satisfied, the results for dynamic systems of our problem is obtained. □

4 Further research and comments

From the well-psedness, stability and dynamics in above sections, we can conclude that the therapy procedure is valid from theory viewpoint. However, there are many factors influencing the therapy process because of the complexity for human body, such as randomness. In the mathematical modeling, the general/degenerate memory and delay on velocity/displacement are also important for the therapy effect, which are interesting topics in application of mathematics.

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