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Ground State Solutions to a Coupled Nonlinear Logarithmic Hartree System

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Abstract. In this paper, we study the following coupled nonlinear logarithmic Hartree system

$$\begin{cases} -\Delta u+\lambda_1 u=\mu_1\left(-\dfrac{1}{2\pi}\ln|x|*u^2\right)u+\beta\left(-\dfrac{1}{2\pi}\ln|x|*v^2\right)u, & x\in\mathbb{R}^2,\\ -\Delta v+\lambda_2 v=\mu_2\left(-\dfrac{1}{2\pi}\ln|x|*v^2\right)v+\beta\left(-\dfrac{1}{2\pi}\ln|x|*u^2\right)v, & x\in\mathbb{R}^2. \end{cases}$$

where β,μ_i,λ_i $(i=1,2)$ are positive constants, $*$ denotes the convolution in $\mathbb{R}^2$. By considering the constraint minimum problem on the Nehari manifold, we prove the existence of ground state solutions for $\beta>0$ large enough. Moreover, we also show that every positive solution is radially symmetric and decays exponentially.

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1 Introduction

The time-dependent system of coupled nonlinear Hartree system can be written as follows:

$$\begin{cases} -i\partial_t \mathbf{\Psi}_1 = \Delta \mathbf{\Psi}_1 + \mu_1 \left(K(x) * |\mathbf{\Psi}_1|^2\right) \mathbf{\Psi}_1 + \beta \left(K(x) * |\mathbf{\Psi}_2|^2\right) \mathbf{\Psi}_1, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^N, \\ -i\partial_t \mathbf{\Psi}_2 = \Delta \mathbf{\Psi}_2 + \mu_2 \left(K(x) * |\mathbf{\Psi}_2|^2\right) \mathbf{\Psi}_2 + \beta \left(K(x) * |\mathbf{\Psi}_1|^2\right) \mathbf{\Psi}_2, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $\mathbf{\Psi}_j : \mathbb{R}^+ \times \mathbb{R}^N \to \mathbb{C}$, i is the imaginary unit, $\mu_1, \mu_2 \neq 0$, and $\beta \neq 0$ is a coupling constant which describes the scattering length of the attractive or repulsive interaction, $K(x)$ is a response function which possesses information on the mutual interaction between the particles. This system (1.1) appears in several physical models, for instance binary mixtures of Bose–Einstein condensates, or the propagation of mutually incoherent wave packets in nonlinear optics (see [1–4]). And if ones want to know more about the physical background and mathematical derivation of Hartree's theory in the case of a single equation, we refer readers to [5,6] and the references therein.

It is well-known that $(\mathbf{\Psi}_1(t,x), \mathbf{\Psi}_2(t,x)) := (e^{i\lambda_1 t} u(x), e^{i\lambda_2 t} v(x))$ is a solitary wave solution of system (1.1) if and only if (u,v) solve the following elliptic system

$$\begin{cases} -\Delta u + \lambda_1 u = \mu_1 \left(K(x) * u^2\right) u + \beta \left(K(x) * v^2\right) u, & \text{in } \mathbb{R}^N, \\ -\Delta v + \lambda_2 v = \mu_2 \left(K(x) * v^2\right) v + \beta \left(K(x) * u^2\right) v, & \text{in } \mathbb{R}^N. \end{cases} \tag{1.2}$$

If the response function is the delta function, i.e., $K(x) = \delta(x)$, then (1.2) turns to the following coupled nonlinear Schrödinger system

$$\begin{cases} -\Delta u + \lambda_1 u = \mu_1 u^3 + \beta v^2 u, & \text{in } \mathbb{R}^N, \\ -\Delta v + \lambda_2 v = \mu_2 v^3 + \beta u^2 v, & \text{in } \mathbb{R}^N. \end{cases} \tag{1.3}$$

For the system (1.3), there are some significant progress on the multiplicity and properties of solutions, see [7–16] and the references therein.

One can see that the fundamental solution to the Laplace operator can be denoted as follows:

$$\Gamma_N(x) = \begin{cases} -\dfrac{1}{2\pi} \ln(|x|), & N = 2; \\ \dfrac{1}{N(N-2) w_N} |x|^{2-N}, & N \geq 3, \end{cases}$$

where w_N is the volume of the unit ball in $\mathbb{R}^N$. If $K(x) = \Gamma_N(x)$ and $N \geq 3$, then system (1.2) can be written as

$$\begin{cases} -\Delta u + \lambda_1 u = \mu_1 \left(\displaystyle\int_{\mathbb{R}^N} \frac{u^2(y)}{|x-y|^{N-2}} \mathrm{d}y \right) u + \beta \left(\displaystyle\int_{\mathbb{R}^N} \frac{v^2(y)}{|x-y|^{N-2}} \mathrm{d}y \right) u, & \text{in } \mathbb{R}^N, \\ -\Delta v + \lambda_2 v = \mu_2 \left(\displaystyle\int_{\mathbb{R}^N} \frac{v^2(y)}{|x-y|^{N-2}} \mathrm{d}y \right) v + \beta \left(\displaystyle\int_{\mathbb{R}^N} \frac{u^2(y)}{|x-y|^{N-2}} \mathrm{d}y \right) v, & \text{in } \mathbb{R}^N, \end{cases} \tag{1.4}$$