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Ground State Solutions to a Coupled Nonlinear Logarithmic Hartree System

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Abstract. In this paper, we study the following coupled nonlinear logarithmic Hartree system

$$\begin{cases} -\Delta u+\lambda_1 u=\mu_1\left(-\dfrac{1}{2\pi}\ln|x|*u^2\right)u+\beta\left(-\dfrac{1}{2\pi}\ln|x|*v^2\right)u, & x\in\mathbb{R}^2,\\ -\Delta v+\lambda_2 v=\mu_2\left(-\dfrac{1}{2\pi}\ln|x|*v^2\right)v+\beta\left(-\dfrac{1}{2\pi}\ln|x|*u^2\right)v, & x\in\mathbb{R}^2. \end{cases}$$

where β,μ_i,λ_i $(i=1,2)$ are positive constants, $*$ denotes the convolution in $\mathbb{R}^2$. By considering the constraint minimum problem on the Nehari manifold, we prove the existence of ground state solutions for $\beta>0$ large enough. Moreover, we also show that every positive solution is radially symmetric and decays exponentially.

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1 Introduction

The time-dependent system of coupled nonlinear Hartree system can be written as follows:

$$\begin{cases} -i\partial_t \mathbf{\Psi}_1 = \Delta \mathbf{\Psi}_1 + \mu_1 \left(K(x)*|\mathbf{\Psi}_1|^2\right)\mathbf{\Psi}_1 + \beta\left(K(x)*|\mathbf{\Psi}_2|^2\right)\mathbf{\Psi}_1, & (t,x)\in\mathbb{R}^+\times\mathbb{R}^N, \\ -i\partial_t \mathbf{\Psi}_2 = \Delta \mathbf{\Psi}_2 + \mu_2 \left(K(x)*|\mathbf{\Psi}_2|^2\right)\mathbf{\Psi}_2 + \beta\left(K(x)*|\mathbf{\Psi}_1|^2\right)\mathbf{\Psi}_2, & (t,x)\in\mathbb{R}^+\times\mathbb{R}^N, \end{cases} \tag{1.1}$$

where $\mathbf{\Psi}_j : \mathbb{R}^+\times\mathbb{R}^N \to \mathbb{C}$, i is the imaginary unit, $\mu_1,\mu_2 \neq 0$, and $\beta \neq 0$ is a coupling constant which describes the scattering length of the attractive or repulsive interaction, $K(x)$ is a response function which possesses information on the mutual interaction between the particles. This system (1.1) appears in several physical models, for instance binary mixtures of Bose–Einstein condensates, or the propagation of mutually incoherent wave packets in nonlinear optics (see [1–4]). And if ones want to know more about the physical background and mathematical derivation of Hartree's theory in the case of a single equation, we refer readers to [5,6] and the references therein.

It is well-known that $(\mathbf{\Psi}_1(t,x),\mathbf{\Psi}_2(t,x)) := (e^{i\lambda_1 t}u(x), e^{i\lambda_2 t}v(x))$ is a solitary wave solution of system (1.1) if and only if (u,v) solve the following elliptic system

$$\begin{cases} -\Delta u + \lambda_1 u = \mu_1\left(K(x)*u^2\right)u + \beta\left(K(x)*v^2\right)u, & \text{in } \mathbb{R}^N, \\ -\Delta v + \lambda_2 v = \mu_2\left(K(x)*v^2\right)v + \beta\left(K(x)*u^2\right)v, & \text{in } \mathbb{R}^N. \end{cases} \tag{1.2}$$

If the response function is the delta function, i.e., $K(x) = \delta(x)$, then (1.2) turns to the following coupled nonlinear Schrödinger system

$$\begin{cases} -\Delta u + \lambda_1 u = \mu_1 u^3 + \beta v^2 u, & \text{in } \mathbb{R}^N, \\ -\Delta v + \lambda_2 v = \mu_2 v^3 + \beta u^2 v, & \text{in } \mathbb{R}^N. \end{cases} \tag{1.3}$$

For the system (1.3), there are some significant progress on the multiplicity and properties of solutions, see [7–16] and the references therein.

One can see that the fundamental solution to the Laplace operator can be denoted as follows:

$$\Gamma_N(x) = \begin{cases} -\dfrac{1}{2\pi}\ln(|x|), & N=2; \\ \dfrac{1}{N(N-2)w_N}|x|^{2-N}, & N\geq 3, \end{cases}$$

where w_N is the volume of the unit ball in $\mathbb{R}^N$. If $K(x)=\Gamma_N(x)$ and $N\geq 3$, then system (1.2) can be written as

$$\begin{cases} -\Delta u + \lambda_1 u = \mu_1 \left(\displaystyle\int_{\mathbb{R}^N} \frac{u^2(y)}{|x-y|^{N-2}}\mathrm{d}y\right)u + \beta\left(\displaystyle\int_{\mathbb{R}^N} \frac{v^2(y)}{|x-y|^{N-2}}\mathrm{d}y\right)u, & \text{in } \mathbb{R}^N, \\ -\Delta v + \lambda_2 v = \mu_2 \left(\displaystyle\int_{\mathbb{R}^N} \frac{v^2(y)}{|x-y|^{N-2}}\mathrm{d}y\right)v + \beta\left(\displaystyle\int_{\mathbb{R}^N} \frac{u^2(y)}{|x-y|^{N-2}}\mathrm{d}y\right)v, & \text{in } \mathbb{R}^N, \end{cases} \tag{1.4}$$

which is a nonlocal problem and has been studied extensively (xw [17–20]).

If $K(x)=\Gamma_N(x)$ and $N=2$, then system (1.2) becomes the following problem

$$\begin{cases} -\Delta u+\lambda_1 u=\mu_1\Big(-\dfrac{1}{2\pi}\ln(|x|)*u^2\Big)u+\beta\Big(-\dfrac{1}{2\pi}\ln(|x|)*v^2\Big)u, & \text{in } \mathbb{R}^2,\\ -\Delta v+\lambda_2 v=\mu_2\Big(-\dfrac{1}{2\pi}\ln(|x|)*v^2\Big)v+\beta\Big(-\dfrac{1}{2\pi}\ln(|x|)*u^2\Big)v, & \text{in } \mathbb{R}^2. \end{cases} \tag{1.5}$$

When $\beta=0$, studying (1.5) is equivalent to studying the following Schrödinger–Poisson system

$$-\Delta u+\lambda u+\mu\Big(\int_{\mathbb{R}^2}\frac{1}{2\pi}\ln(|x-y|)u^2(y)\mathrm{d}y\Big)u=0, \qquad \text{in } \mathbb{R}^2. \tag{1.6}$$

Since the integral kernel $\ln(|x|)$ is sign–changing in $\mathbb{R}^2$, system (1.6) attracts many researchers' attention [21–29]. To study system (1.6), Stubble [29] first set up a variational framework and proved that if $\lambda\geq 0$ and $\mu>0$, then the system (1.6) has a unique ground state solution, which is a positive spherically symmetric decreasing function. Later, Bonheure, Cingolani and Van Schaftingen [22] proved the nondegeneracy and the exponential decay property of the unique ground state solution to (1.6) with $\lambda>0,\mu=1$. Cingolani and Weth [26] considered system (1.6) with a local nonlinear term, i.e.,

$$-\Delta u+\lambda u+\Big(\int_{\mathbb{R}^2}\frac{1}{2\pi}\ln(|x-y|)u^2(y)\mathrm{d}y\Big)u=b|u|^{p-2}, \qquad \text{in } \mathbb{R}^2, \tag{1.7}$$

where $b\geq 0$, $p>2$ and $\lambda\in L^\infty(\mathbb{R}^2)$, and proved that if $p\geq 4$, then the problem (1.7) has a sequence of solution pairs $\pm u$ and a ground sate solution. In addition, the authors also showed that every positive solution is radially symmetric and monotonically decreasing for $p>2$ and $\lambda>0$ by moving plane method. Later on, Du and Weth [27] studied the case of $2<p<4$ and $\lambda=1$, and proved the existence of ground state solutions and infinitely many nontrivial sign–changing solutions. In [24, 25], Chen and Tang considered a more general case related to (1.7) with axially symmetric potential function and general local nonlinearities, and found a ground state solution in the axially symmetric functions space. Recently, Bernini and Mugnai [21] studied the existence of radially symmetric solutions for (1.6) with a local nonlinear term, which does not satisfy the Ambrosetti-Rabinowitz condition.

Motivated by the above mentioned papers, here we want to discuss the existence of positive ground stated solutions and the properties of positive solutions to (1.5) with $\beta,\lambda_i,\mu_i>0(i=1,2)$. The energy functional corresponding to (1.5) is defined by

$$\mathcal{J}(u,v)=\frac{1}{2}\int_{\mathbb{R}^2}\big(|\nabla u|^2+|\nabla v|^2+\lambda_1 u^2+\lambda_2 v^2\big)\,\mathrm{d}x+\frac{1}{4}A_0(u,v), \tag{1.8}$$

where

$$A_0(u,v):=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{1}{2\pi}\ln(|x-y|)\big(\mu_1 u^2(x)u^2(y)+\mu_2 v^2(x)v^2(y)+2\beta u^2(x)v^2(y)\big)\,\mathrm{d}x\mathrm{d}y.$$

Note that $\mathcal{J}$ is not well-defined on $H^1(\mathbb{R}^2)\times H^1(\mathbb{R}^2)$ even if $\beta,\lambda_i,\mu_i>0(i=1,2)$. Inspired by [26,29] , we define a smaller Hilbert space

$$X:=\left\{(u,v)\in H:\int_{\mathbb{R}^2}\big(\ln(1+|x|)u^2+\ln(1+|x|)v^2\big)\,\mathrm{d}x<\infty\right\},\tag{1.9}$$

equipped with the norm

$$\|(u,v)\|_X^2:=\int_{\mathbb{R}^2}\big(|\nabla u|^2+|\nabla v|^2+\lambda_1u^2+\lambda_2v^2\big)\,\mathrm{d}x+\int_{\mathbb{R}^2}\big(\ln(1+|x|)u^2+\ln(1+|x|)v^2\big)\,\mathrm{d}x,$$

where $H:=H^1(\mathbb{R}^2)\times H^1(\mathbb{R}^2)$, endowed with the norm

$$\|(u,v)\|_H^2:=\int_{\mathbb{R}^2}\big(|\nabla u|^2+|\nabla v|^2+\lambda_1u^2+\lambda_2v^2\big)\,\mathrm{d}x.$$

Due to the Hardy-Littlewood-Sobolev inequality and the following decomposition

$$\ln(|x-y|)=\ln(1+|x-y|)-\ln\left(1+\frac{1}{|x-y|}\right),$$

we have that $\mathcal{J}$ is well-defined and of class C^1 on X. Moreover, any critical point of $\mathcal{J}$ in X corresponds to a solution of (1.5).

Before stating our result, we give some definitions. A solution (u,v) of (1.5) is called a nontrivial solution if $u\neq 0$ and $v\neq 0$, and a nontrivial solution (u,v) is positive if $u>0,v>0$. Moreover, we say a solution (u,v) of (1.5) is a ground sate solution if (u,v) is nontrivial and $\mathcal{J}(u,v)\leq\mathcal{J}(\phi,\psi)$ for any other nontrivial solution (ϕ,ψ) of (1.5).

Our first result can be stated as follows:

Theorem 1.1. *Assume that $\beta,\lambda_i,\mu_i>0(i=1,2)$. Then every positive solution $(u,v)\in X$ of* (1.5) *is radially symmetric and monotonically decreasing. In particular, u and v decrease exponentially.*

We note that Wang and Shi [18] showed the radial symmetry and the monotonic decreasing of positive solutions to (1.4) for the case $N=3$ and $\beta,\lambda_i,\mu_i>0(i=1,2)$. Their approach relies on the moving plane method of the integral form. The methods of [18] can apply to a general class of integral equations, but they can not be applied to (1.5) since $\Gamma_2(x)=-\frac{1}{2\pi}\ln(|x|)$ is sign–changing. Inspired by [26], we use a more direct and simpler variant of the moving plane method to prove Theorem 1.1.

To obtain a positive ground state solution of (1.5), we define

$$\mathcal{N}:=\{(u,v)\in X\backslash\{(0,0)\}:N(u,v)=0\},\tag{1.10}$$

$$c:=\inf_{(u,v)\in\mathcal{N}}\mathcal{J}(u,v),\tag{1.11}$$

and

$$\beta_1:=\frac{\mu_1\big(\|\nabla u_1\|_2^2+\lambda_2\|u_1\|_2^2\big)}{\|\nabla u_1\|_2^2+\lambda_1\|u_1\|_2^2},\qquad \beta_2:=\frac{\mu_2\big(\|\nabla u_2\|_2^2+\lambda_1\|u_2\|_2^2\big)}{\|\nabla u_2\|_2^2+\lambda_2\|u_2\|_2^2},\tag{1.12}$$

where

$$N(u,v):=\langle \mathcal{J}'(u,v),(u,v)\rangle=\int_{\mathbb{R}^2}\left(|\nabla u|^2+|\nabla v|^2+\lambda_1 u^2+\lambda_2 v^2\right)\mathrm{d}x+A_0(u,v),$$

$\lambda_i,\mu_i>0(i=1,2)$ and u_i is the unique ground sate solution of (1.6) with $(\lambda,\mu)=(\lambda_i,\mu_i)$ $(i=1,2)$.

Our results on the existence of positive ground state solutions to (1.5) are stated in the following Theorem.

Theorem 1.2. *Assume that $\beta,\lambda_i,\mu_i>0(i=1,2)$. If $\beta>\max\{\beta_1,\beta_2\}$, then system* (1.5) *has a positive ground state solution (u_0,v_0) in X, where β_1 and β_2 are defined in* (1.12).

To obtain a ground sate solution of (1.5), we need to overcome some difficulties. First, since the integral kernel $\Gamma_2(x)=-\frac{1}{2\pi}\ln(|x|)$ is sign–changing, we decompose $\ln(|x|)$ into $\ln(1+|x|)$ and $-\ln(1+\frac{1}{|x|})$. Then, to make the corresponding functional sense, we introduce a new smaller working Space X, which is defined by (1.9). Finally, by studying the constraint minimum problem $c=\inf_{(u,v)\in\mathcal{N}}\mathcal{J}(u,v)$ on the Nehari manifold restricted to X, we obtain the existence of positive ground state solutions of (1.5). Moreover, it is worth noticing that we also need to prove that $\mathcal{N}\in C^1$ is a natural constraint and eliminate the semi–trivial solutions in the processing of proving Theorem 1.2.

The paper is organized as follows. The variational setting and preliminaries will be given in Section 2. Section 3 will devote to the proof of Theorem 1.2. In Section 4, we will complete the proof of Theorem 1.1.

2 Variation framework and preliminaries

For convenience, we introduce the following notations.

- A new Hilbert space

$$X_1:=\left\{u\in H^1(\mathbb{R}^2):\int_{\mathbb{R}^2}\ln(1+|x|)u^2\mathrm{d}x<\infty\right\}$$

  with the norm

$$\|u\|_{X_1}^2:=\int_{\mathbb{R}^2}\left(|\nabla u|^2+u^2\right)\mathrm{d}x+\int_{\mathbb{R}^2}\ln(1+|x|)u^2\mathrm{d}x;$$

- The standard norm in $L^p(\mathbb{R}^2)$ $(1\le p<\infty)$ is denoted by

$$\|u\|_p:=\left(\int_{\mathbb{R}^2}|u|^p\mathrm{d}x\right)^{\frac{1}{p}};$$

- $L^p(\mathbb{R}^2)\times L^p(\mathbb{R}^2)$ $(1\le p<\infty)$ denotes the Lebesgue space with the norm

$$\|(u,v)\|_p:=\left(\|u\|_p^p+\|v\|_p^p\right)^{\frac{1}{p}};$$

- For any $u\in H^1(\mathbb{R}^2)$, $\|u\|_*^2:=\int_{\mathbb{R}^2}\ln(1+|x|)u^2\mathrm{d}x$;

- C, C_1, C_2,$\cdots$ stand for positive constants possibly different in different places.

We define the following symmetric bilinear forms

$$\begin{aligned}
&(u,v)\mapsto I_1(u,v):=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{1}{2\pi}\ln(1+|x-y|)u(x)v(y)\mathrm{d}x\mathrm{d}y;\\
&(u,v)\mapsto I_2(u,v):=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{1}{2\pi}\ln\left(1+\frac{1}{|x-y|}\right)u(x)v(y)\mathrm{d}x\mathrm{d}y;\\
&(u,v)\mapsto I_0(u,v):=I_1(u,v)-I_2(u,v)=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{1}{2\pi}\ln(|x-y|)u(x)v(y)\mathrm{d}x\mathrm{d}y;\\
&(u,v)\mapsto B_i(u,v):=\mu_1I_i(u,u)+\mu_2I_i(v,v)+2\beta I_i(u,v),\qquad i=0,1,2.
\end{aligned}\tag{2.1}$$

Since $0\le\ln(1+r)\le r$ for $r\ge0$, by Hardy-Littlewood-Sobolev inequality (Theorem 4.3 of [30]), we have that

$$I_2(u,v)\le\frac{1}{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{1}{|x-y|}u(x)v(y)\mathrm{d}x\mathrm{d}y\le\mathcal{C}_0\|u\|_{\frac{4}{3}}\|v\|_{\frac{4}{3}},\qquad\forall u,v\in L^{\frac{4}{3}}(\mathbb{R}^2),\tag{2.2}$$

with a constant $\mathcal{C}_0>0$. Also, we define the functionals:

$$\begin{aligned}
&A_1:H^1(\mathbb{R}^2)\times H^1(\mathbb{R}^2)\to[0,\infty], && A_1(u,v):=B_1(u^2,v^2);\\
&A_2:L^{\frac{8}{3}}(\mathbb{R}^2)\times L^{\frac{8}{3}}(\mathbb{R}^2)\to[0,\infty), && A_2(u,v):=B_2(u^2,v^2);\\
&A_0:H^1(\mathbb{R}^2)\times H^1(\mathbb{R}^2)\to\mathbb{R}\cup\{\infty\}, && A_0(u,v):=B_0(u^2,v^2).
\end{aligned}\tag{2.3}$$

From (2.2), we deduce that

$$\begin{aligned}
A_2(u,v)&=\mu_1I_2(u^2,u^2)+\mu_2I_2(v^2,v^2)+2\beta I_2(u^2,v^2)\\
&\le\mathcal{C}_0(\mu_1+\beta)\|u\|_{\frac{8}{3}}^4+\mathcal{C}_0(\mu_2+\beta)\|v\|_{\frac{8}{3}}^4,\qquad\forall(u,v)\in L^{\frac{8}{3}}(\mathbb{R}^2)\times L^{\frac{8}{3}}(\mathbb{R}^2).
\end{aligned}\tag{2.4}$$

Since

$$\ln(1+|x-y|)\le\ln(1+|x|+|y|)\le\ln(1+|x|)+\ln(1+|y|),\qquad\forall x,y\in\mathbb{R}^2,$$

one has

$$I_1(uv,\phi\psi)\le\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{1}{2\pi}\left(\ln(1+|x|)+\ln(1+|y|)\right)|u(x)v(x)||\phi(y)\psi(y)|\mathrm{d}x\mathrm{d}y$$

$$\leq \frac{1}{2\pi}\left(\|u\|_*\|v\|_*\|\phi\|_2\|\psi\|_2+\|u\|_2\|v\|_2\|\phi\|_*\|\psi\|_*\right), \qquad \forall (u,\phi),(v,\psi)\in X, \tag{2.5}$$

which implies that, for any $(u,v)\in X$,

$$\begin{aligned} A_1(u,v)=&\mu_1 I_1(u^2,u^2)+\mu_2 I_1(v^2,v^2)+2\beta I_1(u^2,v^2)\\ &\leq \frac{1}{\pi}\left(\mu_1\|u\|_*^2\|u\|_2^2+\mu_2\|v\|_*^2\|v\|_2^2+\beta\|u\|_*^2\|v\|_2^2+\beta\|v\|_*^2\|u\|_2^2\right). \end{aligned} \tag{2.6}$$

Proposition 2.1 ([31], Gagliardo-Nirenberg inequality). *Let $u\in L^q(\mathbb{R}^N)$ and it is derivatives of order m, $D^m u\in L^r(\mathbb{R}^N)$, $1\leq q,r\leq\infty$. For the derivatives $D^j u$, $0\leq j<m$, the following inequalities hold*

$$\|D^j u\|_p \leq C\|u\|_q^{1-a}\|D^m u\|_r^a, \tag{2.7}$$

where

$$\frac{1}{p}=\frac{j}{N}+a\left(\frac{1}{r}-\frac{m}{N}\right)+(1-a)\frac{1}{q},$$

for all $\frac{j}{m}\leq a\leq 1$ and the constant C depending only on N,m,j,p,r,a.

Lemma 2.1 ([26], Lemma 2.1). *Let $\{u_n\}$ be a sequence in $L^2(\mathbb{R}^2)$ such that $u_n\to u\in L^2(\mathbb{R}^2)\setminus\{0\}$ pointwise a.e. on $\mathbb{R}^2$ and $\{v_n\}$ be a bounded sequence in $L^2(\mathbb{R}^2)$ such that*

$$\sup_{n\in\mathbb{N}} I_1\left(u_n^2,v_n^2\right)<\infty.$$

Then there exists $n_0\in\mathbb{N}$ and $C>0$ such that $\|v_n\|_*<C$ for $n\geq n_0$.

Lemma 2.2. *We have the following properties:*

(i): For all $p\in[2,\infty)$, the embedding $X\hookrightarrow L^p(\mathbb{R}^2)\times L^p(\mathbb{R}^2)$ is compact;

(ii): The functionals A_0,A_1,A_2 and $\mathcal{J}$ are of class C^1 on X. Moreover, for any $(u,v),(\phi,\psi)\in X$,

$$\begin{aligned} \left\langle A_i'(u,v),(\phi,\psi)\right\rangle =&4\mu_1 I_i(u\phi,u^2)+4\mu_2 I_i(v\psi,v^2)\\ &+4\beta I_i(v\psi,u^2)+4\beta I_i(u\phi,v^2), \qquad (i=0,1,2); \end{aligned} \tag{2.8}$$

(iii): $\mathcal{J}$ is weakly lower semicontinuous on X.

Proof. The proof is similar to that of Lemma 2.3 in [23] and Lemma 2.2 in [26], so we omit it. □

Lemma 2.3. *Let $u\in X_1\setminus\{0\}$. Then $w_u\in L^\infty_{loc}(\mathbb{R}^2)$, and*

$$w_u(x)+\frac{1}{2\pi}\|u\|_2^2\ln(|x|)\to 0 \quad \text{as } |x|\to+\infty, \tag{2.9}$$

where

$$w_u(x):=-\int_{\mathbb{R}^2}\frac{1}{2\pi}\ln(|x-y|)u^2(y)\mathrm{d}y.$$

Moreover, if $u\in C^{1,\alpha}_{loc}(\mathbb{R}^2)$ *for any* $0\le\alpha<1$*, then* w_u *is of class* $C^3(\mathbb{R}^2)$ *and satisfies* $-\triangle w_u=u^2$ *in* $\mathbb{R}^2$.

Proof. The proof has been given in [23] or [26]. □

Lemma 2.4. *Assume that* $\beta,\lambda_i,\mu_i>0$ $(i=1,2)$. *If* $(u,v)\in X\setminus\{(0,0)\}$ *is a weak solution of* (1.5). *Then*

(i): $u,v\in C^2(\mathbb{R}^2)$;

(ii): u,v *decay exponentially, i.e., there exist* $C_1>0$ *and* $C_2>0$ *such that*

$$|u(x)|,\ |v(x)|\le C_1e^{-C_2|x|}. \tag{2.10}$$

Proof. We set

$$f_1(x):=\mu_1w_u(x)u(x)+\beta w_v(x)u(x),$$
$$f_2(x):=\mu_2w_v(x)v(x)+\beta w_u(x)v(x),$$

where w_u and w_v are defined in Lemma 2.3. The system (1.5) can be written as

$$\begin{cases}-\Delta u+\lambda_1u=f_1(x), & \text{in } \mathbb{R}^2,\\ -\Delta v+\lambda_2v=f_2(x), & \text{in } \mathbb{R}^2.\end{cases} \tag{2.11}$$

Then, for any bounded open subset $W\subset\subset\mathbb{R}^2$, we find that, for any $p\in[1,\infty)$,

$$\begin{aligned}\|f_1(x)\|^p_{L^p(W)}&\le C_1\int_W\left|w_u^2(x)+2u^2(x)+w_v^2(x)\right|^p\mathrm{d}x\\&\le C_2\left(\|w_u\|^{2p}_{L^\infty(W)}+\|u\|^{2p}_{L^{2p}(W)}+\|w_v\|^{2p}_{L^\infty(W)}\right)\le C,\\ \|f_2(x)\|^p_{L^p(W)}&\le C_2\left(\|w_u\|^{2p}_{L^\infty(W)}+\|v\|^{2p}_{L^{2p}(W)}+\|w_v\|^{2p}_{L^\infty(W)}\right)\le C,\end{aligned} \tag{2.12}$$

since $w_u,w_v\in L^\infty_{loc}(\mathbb{R}^2)$ and $(u,v)\in X\subset H$. Due to (2.12) and the arbitrariness of W, the Interior H^2–Regularity theory implies that $u,v\in W^{2,p}_{loc}(\mathbb{R}^2)$ for any $p\in[1,\infty)$. By Sobolve embedding, we have that $u,v\in C^{1,\alpha}_{loc}(\mathbb{R}^2)$ for any $0\le\alpha<1$, which, together with Lemma 2.3, shows that $w_u,w_v\in C^3(\mathbb{R}^2)$. Then, it is easy to see that f_1,f_2 are locally Hölder continuous. Therefore, $u,v\in C^2(\mathbb{R}^2)$ by elliptic regularity theorem.

It follows from the Agmons Theorem ([32]) that u,v decay exponentially. We complete the proof. □

Lemma 2.5. *Assume that* $\beta,\lambda_i,\mu_i>0$ $(i=1,2)$. *Then*

(i): *For any* $(u,v)\in X\setminus\{(0,0)\}$, *there exists a positive constant* $t_0>0$ *such that*

$$\big(t_0^2u(t_0x),t_0^2v(t_0x)\big)\in\mathcal{N};$$

(ii): *There exists* $\zeta>0$ *such that* $\|(u,v)\|_H\geq\zeta$ *for any* $(u,v)\in\mathcal{N}$;

(iii): $c=\inf_{(u,v)\in\mathcal{N}}\mathcal{J}(u,v)>0$.

Proof. (i): For any $(u,v)\in X\setminus\{(0,0)\}$. Consider

$$\begin{aligned}g(t):=N\big(t^2u(tx),t^2v(tx)\big)&=t^4\big(\|\nabla u\|_2^2+\|\nabla v\|_2^2\big)+t^2\big(\lambda_1\|u\|_2^2+\lambda_2\|v\|_2^2\big)\\&\quad+t^4A_0(u,v)-\frac{t^4\ln t}{2\pi}\Big(\mu_1\|u\|_2^4+\mu_2\|v\|_2^4+2\beta\|u\|_2^2\|v\|_2^2\Big),\qquad t>0.\end{aligned}$$

It is easy to see that $\lim_{t\to0^+}g(t)=0^+$ and $\lim_{t\to+\infty}g(t)=-\infty$. So there exists a positive constant $t_0>0$ such that $N\big(t_0^2u(t_0x),t_0^2v(t_0x)\big)=0$, i.e., $\big(t_0^2u(t_0x),t_0^2v(t_0x)\big)\in\mathcal{N}$.

(ii): We have

$$\begin{aligned}\|(u,v)\|_H^2&=-A_0(u,v)=A_2(u,v)-A_1(u,v)\\&\leq A_2(u,v)\leq C_1\Big(\|u\|_{\frac{8}{3}}^4+\|v\|_{\frac{8}{3}}^4\Big)\leq C\|(u,v)\|_H^4,\qquad\forall(u,v)\in\mathcal{N},\end{aligned}$$

which implies that, for any $(u,v)\in\mathcal{N}$,

$$\|(u,v)\|_H^2\geq\frac{1}{C}=:\zeta^2>0.\tag{2.13}$$

(iii): Using (ii), we get that, for any $(u,v)\in\mathcal{N}$,

$$\mathcal{J}(u,v)=\mathcal{J}(u,v)-\frac{1}{4}N(u,v)=\frac{1}{4}\|(u,v)\|_H^2\geq\frac{1}{4}\zeta^2>0.$$

Hence $c=\inf_{(u,v)\in\mathcal{N}}\mathcal{J}(u,v)\geq\frac{1}{4}\zeta^2>0$. □

Lemma 2.6. *Assume that* $\beta,\lambda_i,\mu_i>0(i=1,2)$. *Then* $\mathcal{N}$ *is a* C^1*–manifold and any critical point of* $\mathcal{J}|_{\mathcal{N}}$ *is a critical point of* $\mathcal{J}$ *in X.*

Proof. Following from Lemma 2.5(i), we have that $\mathcal{N}\neq\varnothing$. Now, we divided our proof into two steps.

(i): By (2.4) and the Sobolev embedding inequality, we find that, for any $r>0$ small enough,

$$\begin{aligned}N(u,v)&=\|(u,v)\|_H^2+A_0(u,v)\geq\|(u,v)\|_H^2-(\mu_1+\beta)\mathcal{C}_0\|u\|_{\frac{8}{3}}^4-(\mu_2+\beta)\mathcal{C}_0\|v\|_{\frac{8}{3}}^4\\&\geq\|(u,v)\|_H^2-C\|(u,v)\|_H^4>0,\qquad\forall\|(u,v)\|_H=r,\end{aligned}$$

which means that $(0,0)\notin\partial\mathcal{N}$.

On the other hand, we can obtain that, for any $(u,v)\in\mathcal{N}$,

$$\langle N'(u,v),(u,v)\rangle=2\|(u,v)\|_H^2+4A_0(u,v)=-2\|(u,v)\|_H^2<0, \tag{2.14}$$

which, together with the Implicit Function Theorem, implies that $\mathcal{N}$ is a C^1–manifold.

(ii): If (u,v) is a critical point of $\mathcal{J}|_{\mathcal{N}}$, i.e., $(u,v)\in\mathcal{N}$ and $(\mathcal{J}|_{\mathcal{N}})'(u,v)=(0,0)$. Then there is a Lagrange multiplier $\gamma\in\mathbb{R}$ such that

$$\mathcal{J}'(u,v)-\gamma N'(u,v)=(0,0). \tag{2.15}$$

Testing (2.15) with (u,v), we get that

$$0=\langle\mathcal{J}'(u,v),(u,v)\rangle-\gamma\langle N'(u,v),(u,v)\rangle=-\gamma\langle N'(u,v),(u,v)\rangle. \tag{2.16}$$

From (2.14) and (2.16), we get that $\gamma=0$. Hence, $\mathcal{J}'(u,v)=(0,0)$, i.e., (u,v) is a critical point of $\mathcal{J}$ in X. The proof of Lemma 2.6 is completed. □

Lemma 2.7. *Assume that $\beta,\lambda_i,\mu_i>0(i=1,2)$. If $\beta>\max\{\beta_1,\beta_2\}$, then we have*

$$c<\min\{\mathcal{J}(u_1,0),\mathcal{J}(0,u_2)\},$$

where β_1 and β_2 are defined in (1.12), *and u_i is the unique ground state solution of* (1.6) *with $(\lambda,\mu)=(\lambda_i,\mu_i)$ $(i=1,2)$.*

Proof. Without loss of generality, we may assume that $\mathcal{J}(u_1,0)\le\mathcal{J}(0,u_2)$. For any $\rho\ge0$, let

$$F_\rho(t):=N(tu_1,t\rho u_1)=t^2\|(u_1,\rho u_1)\|_H^2+t^4(\mu_1+\rho^4\mu_2+2\rho^2\beta)I_0(u_1^2,u_1^2). \tag{2.17}$$

Since u_1 is the unique ground state solution of (1.6) with $(\lambda,\mu)=(\lambda_1,\mu_1)$, one has that

$$\|\nabla u_1\|_2^2+\lambda_1\|u_1\|_2^2+\mu_1 I_0(u_1^2,u_1^2)=0, \tag{2.18}$$

which means that $I_0(u_1^2,u_1^2)<0$. So we can see that

$$t_\rho=\left(\frac{\|(u_1,\rho u_1)\|_H^2}{(\mu_1+\rho^4\mu_2+2\rho^2\beta)\left(-I_0(u_1^2,u_1^2)\right)}\right)^{\frac{1}{2}} \tag{2.19}$$

is the unique positive root of $F_\rho(t)=0$, which implies that $(t_\rho u_1,t_\rho\rho u_1)\in\mathcal{N}$ for any $\rho\ge0$. By (2.18)-(2.19), we can find that, for any $\rho\ge0$,

$$\begin{aligned}h(\rho):=\mathcal{J}(t_\rho u_1,t_\rho\rho u_1)&=\frac{t_\rho^2}{2}\|(u_1,\rho u_1)\|_H^2+\frac{t_\rho^4}{4}\left(\mu_1+\rho^4\mu_2+2\rho^2\beta\right)I_0(u_1^2,u_1^2)\\&=\frac{\|(u_1,\rho u_1)\|_H^4}{4(\mu_1+\rho^4\mu_2+2\rho^2\beta)\left(-I_0(u_1^2,u_1^2)\right)}\end{aligned}$$

$$=\frac{\mu_1\Big(\|\nabla u_1\|_2^2+\lambda_1\|u_1\|_2^2+\rho^2\big(\|\nabla u_1\|_2^2+\lambda_2\|u_1\|_2^2\big)\Big)^2}{4(\mu_1+\rho^4\mu_2+2\rho^2\beta)\big(\|\nabla u_1\|_2^2+\lambda_1\|u_1\|_2^2\big)}$$
$$=\frac{\mu_1(b_1+\rho^2 b_2)^2}{4(\mu_1+\rho^4\mu_2+2\rho^2\beta)b_1},$$

where $b_i:=\|\nabla u_1\|_2^2+\lambda_i\|u_1\|_2^2$ $(i=1,2)$. Since $\beta>\beta_1$, we have $b_2\mu_1-b_1\beta<0$, which, together with the derivative of $h(\rho)$, gives that

$$h'(\rho)=\frac{\mu_1\Big[4\big(b_1+\rho^2b_2\big)\rho b_2\big(\mu_1+\rho^4\mu_2+2\rho^2\beta\big)-\big(4\rho^3\mu_2+4\rho\beta\big)\big(b_1+\rho^2b_2\big)^2\Big]}{4b_1\big(\mu_1+\rho^4\mu_2+2\rho^2\beta\big)^2}$$
$$=\frac{\mu_1\Big[\big(4b_1b_2\mu_1-4b_1^2\beta\big)\rho+o(\rho^2)\Big]}{4b_1\big(\mu_1+\rho^4\mu_2+2\rho^2\beta\big)^2}\to 0^-,\qquad \text{as } \rho\to 0^+. \tag{2.20}$$

So there exists $\rho_1>0$ such that

$$c\le\mathcal{J}(t_{\rho_1}u_1,t_{\rho_1}\rho_1u_1)=h(\rho_1)<h(0)=\frac{1}{4}b_1=\frac{1}{4}\big(\|\nabla u_1\|_2^2+\lambda_1\|u_1\|_2^2\big)$$
$$=\frac{1}{2}\big(\|\nabla u_1\|_2^2+\lambda_1\|u_1\|_2^2\big)+\frac{1}{4}\mu_1I_0(u_1^2,u_1^2)=\mathcal{J}(u_1,0)=\min\{\mathcal{J}(u_1,0),\mathcal{J}(0,u_2)\}. \tag{2.21}$$

We complete the proof. □

3 The proof of Theorem 1.2

Proof of Theorem 1.2*:* Assume that $\{(u_n,v_n)\}\subset\mathcal{N}$ such that $\mathcal{J}(u_n,v_n)\to c$ as $n\to\infty$. It is easy to see that

$$c+o(1)=\mathcal{J}(u_n,v_n)-\frac{1}{4}N(u_n,v_n)=\frac{1}{4}\|(u_n,v_n)\|_H^2,$$

which tells us that $\{(u_n,v_n)\}$ is bounded in H. Using (2.7), we have $\|u\|_{\frac{8}{3}}^4\le C\|u\|_2^3\|\nabla u\|_2$ for any $u\in H^1(\mathbb{R}^2)$, which, together with (2.4), means that

$$0\le A_2(u_n,v_n)\le\mathcal{C}_0(\mu_1+\beta)\|u_n\|_{\frac{8}{3}}^4+\mathcal{C}_0(\mu_2+\beta)\|v_n\|_{\frac{8}{3}}^4$$
$$\le C_1\big(\|u_n\|_2^3\|\nabla u_n\|_2+\|v_n\|_2^3\|\nabla v_n\|_2\big)\le C. \tag{3.1}$$

So we can get that

$$c+o(1)=\mathcal{J}(u_n,v_n)=\frac{1}{2}\|(u_n,v_n)\|_H^2+\frac{1}{4}A_0(u_n,v_n)$$

$$\geq \frac{1}{4}A_1(u_n,v_n)-\frac{1}{4}A_2(u_n,v_n)\geq \frac{1}{4}\big(\mu_1 I_1(u_n^2,u_n^2)+\mu_2 I_1(v_n^2,v_n^2)+2\beta I_1(u_n^2,v_n^2)\big)-C,$$

which implies that $\{I_1(u_n^2,v_n^2)\}$ is bounded. Since $\{I_1(u_n^2,v_n^2)\}$ and $\{\|(u_n,v_n)\|_H^2\}$ are bounded in $\mathbb{R}$, it follows from Lemma 2.1 that $\{\|u_n\|_*^2+\|v_n\|_*^2\}$ is bounded. So $\{(u_n,v_n)\}$ is bounded in X. Passing to a subsequence, one has that

$$\begin{aligned}&(u_n,v_n)\rightharpoonup(u_0,v_0)\ \text{ in } X,\\&(u_n,v_n)\to(u_0,v_0)\ \text{ in } L^p(\mathbb{R}^2)\times L^p(\mathbb{R}^2)\ \text{ for } p\in[2,\infty),\\&(u_n,v_n)\to(u_0,v_0)\ \text{ a.e. on } \mathbb{R}^2.\end{aligned}\tag{3.2}$$

By the definition of A_0,A_1,A_2 and (3.1), we find

$$\begin{aligned}c+o(1)&=\mathcal{J}(u_n,v_n)-\frac{1}{2}N(u_n,v_n)=-\frac{1}{4}A_0(u_n,v_n)\\&\leq\frac{1}{4}A_2(u_n,v_n)\leq C_1\big(\|u_n\|_2^3\|\nabla u_n\|_2+\|v_n\|_2^3\|\nabla v_n\|_2\big)\\&\leq C(\|u_n\|_2+\|v_n\|_2)^3(\|\nabla u_n\|_2+\|\nabla v_n\|_2)\\&\leq C(\|u_n\|_2+\|v_n\|_2)^3,\end{aligned}$$

which shows that $\|u_n\|_2+\|v_n\|_2>0$. Using (3.2), we get that $(u_0,v_0)\neq(0,0)$. Passing to $(|u_n|,|v_n|)$, we may assume that (u_0,v_0) is nonnegative.

By the weak lower semicontinuity of norm and $\mathcal{J}$ in X, we can conclude that $N(u_0,v_0)\leq \liminf_{n\to\infty}N(u_n,v_n)=0$, which, together with $\lim_{t\to 0^+}N(t^2u_0(tx),t^2v_0(tx))=0^+$, implies that there exists $0<t_0\leq 1$ such that $(t_0^2u_0(t_0x),t_0^2v_0(t_0x))\in\mathcal{N}$. Therefore, we find that

$$\begin{aligned}c&=\lim_{n\to\infty}\mathcal{J}(u_n,v_n)=\lim_{n\to\infty}\big[\mathcal{J}(u_n,v_n)-\frac{1}{4}N(u_n,v_n)\big]=\lim_{n\to\infty}\frac{1}{4}\|(u_n,v_n)\|_H^2\\&\geq\frac{1}{4}\|(u_0,v_0)\|_H^2\geq\frac{t_0^4}{4}(\|\nabla u_0\|_2^2+\|\nabla v_0\|_2^2)+\frac{t_0^2}{4}(\lambda_1\|u_0\|_2^2+\lambda_2\|v_0\|_2^2)\\&=\frac{1}{4}\big\|\big(t_0^2u_0(t_0x),t_0^2v_0(t_0x)\big)\big\|_H^2=\mathcal{J}\big(t_0^2u_0(t_0x),t_0^2v_0(t_0x)\big)\geq c,\end{aligned}$$

which gives that $t_0=1$. So $\mathcal{J}(u_0,v_0)=c$ and $(u_0,v_0)\in\mathcal{N}$, i.e., c is achieved by (u_0,v_0). From Lemma 2.7, we get that

$$J(u_0,v_0)=c<\min\{\mathcal{J}(u_1,0),\mathcal{J}(0,u_2)\}.$$

Using Lemmas 2.4 and 2.6, we find that $(u_0,v_0)\in C^2(\mathbb{R}^2)\times C^2(\mathbb{R}^2)$ is a nonnegative nontrivial ground state solution of (1.5). Combining the following decomposition

$$\begin{aligned}&\int_{\mathbb{R}^2}\ln(|x-y|)u^2(y)\mathrm{d}y\\=&\int_{\mathbb{R}^2}\ln(1+|x-y|)u^2(y)\mathrm{d}y-\int_{\mathbb{R}^2}\ln\left(1+\frac{1}{|x-y|}\right)u^2(y)\mathrm{d}y,\qquad\forall u\in X_1\end{aligned}$$

and the strong maximum principle, we have that (u_0,v_0) is a positive ground state solution of (1.5). The proof of Theorem 1.2 is completed. □

4 The proof of Theorem 1.1

In this section, for any $t\in\mathbb{R}$, we let

$$H_t:=\{x=(x_1,x_2)\in\mathbb{R}^2:x_1>t\},\qquad T_t:=\{x=(x_1,x_2)\in\mathbb{R}^2:x_1=t\},$$

and

$$x^t:=(2t-x_1,x_2),\qquad \forall x=(x_1,x_2)\in\mathbb{R}^2.$$

Assume that (u,v) is a fixed positive solution of (1.5) in X and set

$$u^t(x):=u(x^t),\quad v^t(x):=v(x^t),\quad w_u^t(x):=w_u(x^t),\quad w_v^t(x):=w_v(x^t),\qquad x\in\mathbb{R}^2,\ t\in\mathbb{R},$$

and

$$u_t:=u^t-u,\quad v_t:=v^t-v,\quad w_{u,t}:=w_u^t-w_u,\quad w_{v,t}:=w_v^t-w_v,\ \text{ in } H_t,$$

where w_u,w_v are defined in Lemma 2.3. By direct computation, we have that

$$\begin{cases}-\Delta u_t+\lambda_1u_t=\mu_1w_{u,t}u^t+\mu_1w_uu_t+\beta w_{v,t}u^t+\beta w_vu_t, & \text{in } H_t,\\ -\Delta v_t+\lambda_2v_t=\mu_2w_{v,t}v^t+\mu_2w_vv_t+\beta w_{u,t}v^t+\beta w_uv_t, & \text{in } H_t,\\ -\Delta w_{u,t}=(u^t)^2-u^2=(u^t+u)u_t, & \text{in } H_t,\\ -\Delta w_{v,t}=(v^t)^2-v^2=(v^t+v)v_t, & \text{in } H_t.\end{cases}\tag{4.1}$$

Lemma 4.1. *Assume that (u,v) is a positive solution of* (1.5) *and w_u,w_v are defined in Lemma* 2.3. *Then we have,*

$$\begin{aligned}w_{u,t}(x)=w_u^t-w_u=\int_{H_t}\frac{1}{2\pi}\ln\left(\frac{|x-y^t|}{|x-y|}\right)\left(u^t(y)+u(y)\right)u_t(y)\mathrm{d}y,\qquad x\in\mathbb{R}^2,\\ w_{v,t}(x)=w_v^t-w_v=\int_{H_t}\frac{1}{2\pi}\ln\left(\frac{|x-y^t|}{|x-y|}\right)\left(v^t(y)+v(y)\right)v_t(y)\mathrm{d}y,\qquad x\in\mathbb{R}^2.\end{aligned}\tag{4.2}$$

Proof. Since $|x^t-y^t|=|x-y|$ and $|x^t-y|=|x-y^t|$, we find that

$$\begin{aligned}w_{u,t}(x)=&w_u^t-w_u=-\int_{\mathbb{R}^2}\frac{1}{2\pi}\ln(|x^t-y|)u^2(y)\mathrm{d}y+\int_{\mathbb{R}^2}\frac{1}{2\pi}\ln(|x-y|)u^2(y)\mathrm{d}y\\ =&-\int_{H_t}\frac{1}{2\pi}\ln(|x^t-y|)u^2(y)\mathrm{d}y-\int_{\mathbb{R}^2\setminus H_t}\frac{1}{2\pi}\ln(|x^t-y|)u^2(y)\mathrm{d}y\\ &+\int_{H_t}\frac{1}{2\pi}\ln(|x-y|)u^2(y)\mathrm{d}y+\int_{\mathbb{R}^2\setminus H_t}\frac{1}{2\pi}\ln(|x-y|)u^2(y)\mathrm{d}y\\ =&-\int_{H_t}\frac{1}{2\pi}\ln(|x-y^t|)u^2(y)\mathrm{d}y-\int_{H_t}\frac{1}{2\pi}\ln(|x^t-y^t|)u^2(y^t)\mathrm{d}y\\ &+\int_{H_t}\frac{1}{2\pi}\ln(|x-y|)u^2(y)\mathrm{d}y+\int_{H_t}\frac{1}{2\pi}\ln(|x-y^t|)u^2(y^t)\mathrm{d}y\end{aligned}$$

$$
\begin{aligned}
&=\int_{H_t}\frac{1}{2\pi}\left(\ln(|x-y^t|)-\ln(|x-y|)\right)\left(u^2(y^t)-u^2(y)\right)\mathrm{d}y\\
&=\int_{H_t}\frac{1}{2\pi}\ln\left(\frac{|x-y^t|}{|x-y|}\right)\left(u^t(y)+u(y)\right)u_t(y)\mathrm{d}y.
\end{aligned}
$$

Similarly, one has that

$$
w_{v,t}(x)=\int_{H_t}\frac{1}{2\pi}\ln\left(\frac{|x-y^t|}{|x-y|}\right)\left(v^t(y)+v(y)\right)v_t(y)\mathrm{d}y.
$$

We complete the proof. □

Lemma 4.2 ([26], Lemma 6.2). *There exists a constant $k>0$ such that*

$$
\|w_{u,t}^-\|_{L^2(H_t)}\le kc_{u,t}\|u_t^-\|_{L^2(H_t)},\qquad \text{for every } t\in\mathbb{R}, \tag{4.3}
$$

where

$$
c_{u,t}=\left(\int_{H_t^u}(y_1-t)^2u^2(y)\mathrm{d}y\right)^{\frac{1}{2}},\qquad H_t^u:=\{x\in H_t: u_t(x)<0\}, \tag{4.4}
$$

and $h^-:=\min\{h,0\}$ for any $h\in X_1$.

Lemma 4.3. *Assume that $\beta,\lambda_i,\mu_i>0(i=1,2)$. There exists $T>0$ such that, when $t\ge T$, $u_t,v_t\ge 0$ in H_t.*

Proof. From (2.9), we can choose $T_1>0$ such that $w_u,w_v\le 0$ in H_t for every $t>T_1$. By (4.1) and (4.3), we have

$$
\begin{aligned}
&\|u_t^-\|_{L^2(H_t)}^2+\|v_t^-\|_{L^2(H_t)}^2\le\|(u_t^-,v_t^-)\|_{H^1(H_t)\times H^1(H_t)}^2\\
=&\int_{H_t}\left(\mu_1 w_{u,t}u^tu_t^-+\mu_1 w_u(u_t^-)^2+\beta w_{v,t}u^tu_t^-+\beta w_v(u_t^-)^2\right)\mathrm{d}x\\
&\quad+\int_{H_t}\left(\mu_2 w_{v,t}v^tv_t^-+\mu_2 w_v(v_t^-)^2+\beta w_{u,t}v^tv_t^-+\beta w_u(v_t^-)^2\right)\mathrm{d}x\\
\le&\int_{H_t}\left(\mu_1 w_{u,t}^-u^tu_t^-+\beta w_{v,t}^-u^tu_t^-+\mu_2 w_{v,t}^-v^tv_t^-+\beta w_{u,t}^-v^tv_t^-\right)\mathrm{d}x\\
\le& C_1\|w_{u,t}^-\|_{L^2(H_t)}\left(\|u^t\|_{L^\infty(H_t)}\|u_t^-\|_{L^2(H_t)}+\|v^t\|_{L^\infty(H_t)}\|v_t^-\|_{L^2(H_t)}\right)\\
&\quad+C_1\|w_{v,t}^-\|_{L^2(H_t)}\left(\|u^t\|_{L^\infty(H_t)}\|u_t^-\|_{L^2(H_t)}+\|v^t\|_{L^\infty(H_t)}\|v_t^-\|_{L^2(H_t)}\right)\\
\le& C_2\left(c_{u,t}\|u_t^-\|_{L^2(H_t)}+c_{v,t}\|v_t^-\|_{L^2(H_t)}\right)\left(\|u_t^-\|_{L^2(H_t)}+\|v_t^-\|_{L^2(H_t)}\right)\\
\le& C\left(c_{u,t}+c_{v,t}\right)\left(\|u_t^-\|_{L^2(H_t)}^2+\|v_t^-\|_{L^2(H_t)}^2\right).
\end{aligned} \tag{4.5}
$$

By the definitions of $c_{u,t}, c_{v,t}$ and the fact that u,v decay exponentially, we find that

$$\lim_{t\to\infty}(c_{u,t}+c_{v,t})=0,$$

which, together with (4.5), implies that there exists $T>T_1$ such that, for any $t\geq T$,

$$u_t^-\equiv 0,\ v_t^-\equiv 0,\qquad x\in H_t.$$

We complete the proof. □

Lemma 4.4. *Assume that* $\beta,\lambda_i,\mu_i>0$ $(i=1,2)$. *Let* $t\in\mathbb{R}$ *and* $u_t\geq 0, v_t\geq 0$ *in* H_t. *Then*
(i): $w_{u,t}\geq 0, w_{v,t}\geq 0$ *in* H_t;
(ii): *If* $u_t\not\equiv 0$ *or* $v_t\not\equiv 0$, *then we have that*

$$u_t>0,\ v_t>0 \quad \text{in } H_t \quad \text{and} \quad \frac{\partial u}{\partial x_1}<0,\quad \frac{\partial v}{\partial x_1}<0 \text{ in } T_t. \tag{4.6}$$

Proof. (i): Since $\ln\left(\frac{|x-y^t|}{|x-y|}\right)>0$ for every $x,y\in H_t$, by Lemma 4.1, we find that, for every $t\in\mathbb{R}$,

$$w_{u,t},\ w_{v,t}\geq 0 \ \text{ in } H_t. \tag{4.7}$$

(ii): Without loss of generality, we assume that $u_t\not\equiv 0$ in H_t. Then we get that $w_{u,t}>0$ in H_t by (4.2). So, (4.1) and (4.7) imply that

$$-\Delta u_t+\left(\lambda_1-\mu_1 w_u^- -\beta w_v^-\right)u_t=(\mu_1 w_{u,t}+\beta w_{v,t})u^t+\left(\mu_1 w_u^+ +\beta w_v^+\right)u_t>0,\quad \text{in } H_t,$$

and

$$-\Delta v_t+\left(\lambda_2-\mu_2 w_v^- -\beta w_u^-\right)v_t=(\mu_2 w_{v,t}+\beta w_{u,t})v^t+\left(\mu_2 w_v^+ +\beta w_u^+\right)v_t>0,\quad \text{in } H_t,$$

where $h^+:=\max\{h,0\}$ for any $h\in X$. Hence $u_t>0$, $v_t>0$ in H_t by the maximum principle, and

$$-2\frac{\partial u}{\partial x_1}=\frac{\partial u_t}{\partial x_1}>0,\qquad -2\frac{\partial v}{\partial x_1}=\frac{\partial v_t}{\partial x_1}>0 \ \text{ in } T_t,$$

by the Hopf Lemma. We complete the proof. □

Lemma 4.5. *Assume that* $\beta,\lambda_i,\mu_i>0 (i=1,2)$. *Let* $u_t(x), v_t(x)\geq 0$ *in* H_t, *but* $u_t\not\equiv 0$ *or* $v_t\not\equiv 0$ *in* H_t. *Then there exists* $\varepsilon>0$ *such that, for any* $\tau\in(t-\varepsilon,t]$, $u_\tau\geq 0, v_\tau\geq 0$ *in* H_τ.

Proof. Let $B_R:=B_R(0)$ for $R>0$. By (2.9) and the fact that u,v decay exponentially, we may choose $R>1$ large enough such that, for every $\tau\in\mathbb{R}$,

$$w_u\leq 0,\ w_v\leq 0 \quad \text{in } H_\tau\backslash B_R, \tag{4.8}$$

and for any $\tau\in[t-1,t]$,

$$\left(\int_{\mathbb{R}^2\backslash B_R}(y_1-\tau)^2u^2(y)\mathrm{d}y\right)^{\frac{1}{2}}+\left(\int_{\mathbb{R}^2\backslash B_R}(y_1-\tau)^2v^2(y)\mathrm{d}y\right)^{\frac{1}{2}}<\frac{1}{2C_0}, \tag{4.9}$$

where $C_0:=k(\mu_1+\mu_2+2\beta)(\|u^\tau\|_{L^\infty(H_\tau)}+\|v^\tau\|_{L^\infty(H_\tau)})$ and k is defined in Lemma 4.2. By Lemma 4.4(ii) and $u,v\in C^2(\mathbb{R}^2)$, there exists $0<\varepsilon<1$ such that, for any $\tau\in(t-\varepsilon,t]$,

$$u_\tau>0,\ v_\tau>0\ \text{ in } H_\tau\cap B_R, \tag{4.10}$$

which means that

$$u_\tau^-=0,\ v_\tau^-=0\ \text{ in } H_\tau\cap B_R. \tag{4.11}$$

Using (4.1), (4.3), (4.8) and (4.11), we can get that

$$\begin{aligned}
&\|u_\tau^-\|^2_{L^2(H_\tau)}+\|v_\tau^-\|^2_{L^2(H_\tau)}\le\|(u_\tau^-,v_\tau^-)\|^2_{H^1(H_\tau)\times H^1(H_\tau)}\\
=&\int_{H_\tau\backslash B_R}\left(\mu_1w_{u,\tau}u^\tau u_\tau^-+\mu_1w_u(u_\tau^-)^2+\beta w_{v,\tau}u^\tau u_\tau^-+\beta w_v(u_\tau^-)^2\right)\mathrm{d}x\\
&\quad+\int_{H_\tau\backslash B_R}\left(\mu_2w_{v,\tau}v^\tau v_\tau^-+\mu_2w_v(v_\tau^-)^2+\beta w_{u,\tau}v^\tau v_\tau^-+\beta w_u(v_\tau^-)^2\right)\mathrm{d}x\\
\le&\int_{H_\tau\backslash B_R}\left(\mu_1w_{u,\tau}^-u^\tau u_\tau^-+\beta w_{v,\tau}^-u^\tau u_\tau^-+\mu_2w_{v,\tau}^-v^\tau v_\tau^-+\beta w_{u,\tau}^-v^\tau v_\tau^-\right)\mathrm{d}x\\
\le&\|w_{u,\tau}^-\|_{L^2(H_\tau)}\left(\mu_1\|u^\tau\|_{L^\infty(H_\tau)}\|u_\tau^-\|_{L^2(H_\tau)}+\beta\|v^\tau\|_{L^\infty(H_\tau)}\|v_\tau^-\|_{L^2(H_\tau)}\right)\\
&\quad+\|w_{v,\tau}^-\|_{L^2(H_\tau)}\left(\beta\|u^\tau\|_{L^\infty(H_\tau)}\|u_\tau^-\|_{L^2(H_\tau)}+\mu_2\|v^\tau\|_{L^\infty(H_\tau)}\|v_\tau^-\|_{L^2(H_\tau)}\right)\\
\le&kc_{u,\tau}\|u_\tau^-\|_{L^2(H_\tau)}\left(\mu_1\|u^\tau\|_{L^\infty(H_\tau)}\|u_\tau^-\|_{L^2(H_\tau)}+\beta\|v^\tau\|_{L^\infty(H_\tau)}\|v_\tau^-\|_{L^2(H_\tau)}\right)\\
&\quad+kc_{v,\tau}\|v_\tau^-\|_{L^2(H_\tau)}\left(\beta\|u^\tau\|_{L^\infty(H_\tau)}\|u_\tau^-\|_{L^2(H_\tau)}+\mu_2\|v^\tau\|_{L^\infty(H_\tau)}\|v_\tau^-\|_{L^2(H_\tau)}\right)\\
\le&k\left(\mu_1c_{u,\tau}\|u^\tau\|_{L^\infty(H_\tau)}+\beta c_{u,\tau}\|v^\tau\|_{L^\infty(H_\tau)}+\beta c_{v,\tau}\|u^\tau\|_{L^\infty(H_\tau)}\right)\|u_\tau^-\|^2_{L^2(H_\tau)}\\
&\quad+k\left(\beta c_{u,\tau}\|v^\tau\|_{L^\infty(H_\tau)}+\beta c_{v,\tau}\|u^\tau\|_{L^\infty(H_\tau)}+\mu_2c_{v,\tau}\|v^\tau\|_{L^\infty(H_\tau)}\right)\|v_\tau^-\|^2_{L^2(H_\tau)}\\
\le&C_0(c_{u,\tau}+c_{v,\tau})\left(\|u_\tau^-\|^2_{L^2(H_\tau)}+\|v_\tau^-\|^2_{L^2(H_\tau)}\right).
\end{aligned} \tag{4.12}$$

Using (4.10) and the definition of H_τ^w, we have $H_\tau^u,H_\tau^v\subset\mathbb{R}^2\backslash B_R$, which, combining (4.9) and the definition of $c_{w,\tau}$, means that

$$c_{u,\tau}+c_{v,\tau}\le\left(\int_{\mathbb{R}^2\backslash B_R}(y_1-\tau)^2u^2(y)\mathrm{d}y\right)^{\frac{1}{2}}+\left(\int_{\mathbb{R}^2\backslash B_R}(y_1-\tau)^2v^2(y)\mathrm{d}y\right)^{\frac{1}{2}}<\frac{1}{2C_0}.$$

So $\|u_\tau^-\|_{L^2(H_\tau)}^2+\|v_\tau^-\|_{L^2(H_\tau)}^2=0$, which implies that $u_\tau^-\equiv 0$, $v_\tau^-\equiv 0$ in H_τ for any $\tau\in(t-\varepsilon,t]$. We complete the proof. □

Proof of Theorem 1.1*:* From Lemma 4.3, there exists $T>0$ such that, for any $t>T$,

$$u_t(x)\geq 0, \quad v_t(x)\geq 0 \ \text{ in } H_t. \tag{4.13}$$

Starting from such a $t>T$, one can move the plane $x_1=t$ to the left as long as (4.13) holds. Suppose that there exists a $t_0>0$ such that $u_{t_0}(x),v_{t_0}(x)\geq 0$ in H_{t_0}, but $u_{t_0}\not\equiv 0$ or $v_{t_0}\not\equiv 0$ in H_{t_0}. By Lemma 4.5, there exists a $\varepsilon>0$ such that, for any $\tau\in(t_0-\varepsilon,t_0]$,

$$u_\tau(x)\geq 0, \quad v_\tau(x)\geq 0 \ \text{ in } H_\tau.$$

Using Lemma 4.4(ii), we have $u_\tau\equiv 0$ in H_τ if and only if $v_\tau\equiv 0$ in H_τ. So we obtain that if the process of moving plane stops at t_1, then $u_{t_1}\equiv 0,v_{t_1}\equiv 0$ in H_{t_1} and $u_t\geq 0,v_t\geq 0$ in H_t for any $t\geq t_1$.
By a translation, we may assume that $u(0)=\max_{x\in\mathbb{R}^2}u(x)$ and $v(0)=\max_{x\in\mathbb{R}^2}v(x)$. Therefore, the process of moving plane from any direction must stop at the origin. So u and v are radially symmetric and monotone decreasing. We complete the proof. □

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References

[1] Dalfovo F., Giorgini S., Pitaevskii L. P. and Stringari S., Theory of Bose-Einstein condensation in trapped gases. *Rev. Modern Phys.*, **71** (3) (1999), 463-512.
[2] Esry B. D., Hartree-Fock theory for Bose-Einstein condensates and the inclusion of correlation effects. *Phys. Rev. A.*, **55** (2) (1997), 1147-1159.
[3] Mitchell M., Segev M., Self-trapping of incoherent white light. *Nature.*, **387** (6636) (1997), 880-883.
[4] Vowe S., Lämmerzahl C. and Krutzik M., Detecting a logarithmic nonlinearity in the Schrödinger equation using Bose-Einstein condensates. *Phys. Rev. A.*, **101** (4) (2020), 043617.
[5] Lewin M., Nam P. and Rougerie N., The mean-field approximation and the non-linear Schrödinger functional for trapped Bose gases. *Trans. Amer. Math. Soc.*, **368** (9) (2016), 6131-6157.
[6] Lewin M., Mean-field limit of Bose systems: rigorous results. *preprint.*, 2015, arXiv:1510.044 07.
[7] Ambrosetti A., Colorado E., Standing waves of some coupled nonlinear Schrödinger equations. *J. Lond. Math. Soc.*, **75** (2007), 67-82.

[8] Bartsch T., Wang Z. Q., Note on ground states of nonlinear Schrödinger systems. *J. Partial Differential Equations*., **19** (2006), 200-207.

[9] Chang J. Y., Liu Z. L., Ground states of nonlinear Schrödinger systems. *Proc. Amer. Math. Soc.*, **138** (2) (2010), 687-693.

[10] Chen Z. J., Zou W. M., An optimal constant for the existence of least energy solutions of a coupled Schrödinger system. *Calc. Var. Partial Differential Equations*., **48** (2013), 695-711.

[11] Chen Z. J., Zou W. M., Positive least energy solutions and phase separation for coupled Schrödinger equations with critical exponent. *Arch. Ration. Mech. Anal.*, **205** (2) (2012), 515-551.

[12] Liu H. D., Liu Z. L., Ground states of a nonlinear Schrödinger system with nonconstant potentials. *Science China Mathematics*., **58** (2) (2015), 257-278.

[13] Lin T. C., Wei J. C., Ground state of N coupled nonlinear Schrödinger equations in $\mathbb{R}^n$, $n \leq 3$. *Comm. Math. Phys.*, **255** (3) (2005), 629-653.

[14] Liu Z. L., Wang Z. Q., Ground states and bound states of a nonlinear Schrödinger system. *Adv. Nonlinear Stud.*, **10** (1) (2010), 175-193.

[15] Ma L., Zhao L., Uniqueness of ground states of some coupled nonlinear Schrödinger systems and their application. *J. Differential Equations*., **245** (9) (2008), 2551-2565.

[16] Sirakov B., Least energy solitary waves for a system of nonlinear Schrödinger equations in $\mathbb{R}^N$. *Comm. Math. Phys.*, **271** (1) (2007), 199-221.

[17] Wang J., Wei S., Existence of ground state solutions of the coupled nonlinear Hartree type system with attractive couplings. *SN Partial Differ. Equ. Appl.*, **56** (1) (2021), 1-16.

[18] Wang J., Shi J. P., Standing waves for a coupled nonlinear Hartree equations with nonlocal interaction. *Calc. Var. Partial Differential Equations*., **56** (6) (2017), 1-36.

[19] Yang M. B., Wei Y. H. and Ding Y. H., Existence of semiclassical states for a coupled Schrödinger system with potentials and nonlocal nonlinearities. *Z. Angew. Math. Phys.*, **65** (1) (2014), 41-68.

[20] You S., Zhao P. H. and Wang Q. X., Existence and asymptotic behavior of positive least energy solutions for coupled nonlinear Choquard equations. *Electron. J. Differential Equations*., **47** (2021), 1-20.

[21] Bernini F., Mugnai D., On a logarithmic Hartree equation. *Adv. Nonlinear Anal.*, **9** (1) (2020), 850-865.

[22] Bonheure D., Cingolani S. and Van Schaftingen J., The logarithmic Choquard equation: Sharp asymptotics and nondegeneracy of the groundstate. *J. Funct. Anal.*, **272** (12) (2017), 5255-5281.

[23] Cao D. M., Dai W. and Zhang Y., Existence and symmetry of solutions to 2-D Schrödinger-Newton equations. *Dyn. Partial Differ. Equ.*, **18** (2) (2021), 113-156.

[24] Chen S. T., Tang X. H., Existence of ground state solutions for the planar axially symmetric Schrödinger-Poisson system. *Discrete Contin. Dyn. Syst. Ser. B*., **24** (9) (2019), 4685.

[25] Chen S. T., Tang X. H., On the planar Schrödinger-Poisson system with the axially symmetric potential. *J. Differential Equations*., **268** (3) (2020), 945-976.

[26] Cingolani S., Weth T., On the planar Schrödinger-Poisson system. *Ann. Inst. H. Poincaré Anal. Non Linéaire*., **33** (1) (2016), 169-197.

[27] Du M., Weth T., Ground states and high energy solutions of the planar Schrödinger-Poisson system. *Nonlinearity*., **30** (9) (2017), 3492-3515.

[28] Liu Z. S., Rădulescu V. D., Tang C. L. and et al., Another look at planar Schrödinger-Newton systems. *J. Differential Equations*, **328** (2022), 65-104.

[29] Stubbe J., Bound states of two-dimensional Schrödinger-Newton equations. *preprint.*, 2008, arXiv:0807.4059.
[30] Lieb E. H., Loss M., Analysis, American Mathematical Society, Providence, 2001.
[31] Nirenberg L., On elliptic partial differential equations. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.*, **13** (1) (1959), 115-162.
[32] Agmon S., Lectures on Exponential Decay of Solutions of Second-order Elliptic Equations: Bounds on Eigenfunctions of N-Body Schrödinger Operators. Math. Notes, Vol. 29, Princeton University Press, New Jersey(NJ), 1982.