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On a Nonhomogeneous N-Laplacian Problem with Double Exponential Critical Growth

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Abstract. This paper is devoted to study the existence and multiplicity of nontrivial solutions for the following boundary value problem

$$\begin{cases} -\mathrm{div}\Big(\omega(x)|\nabla u(x)|^{N-2}\nabla u(x)\Big)=f(x,u)+\epsilon h(x), & \text{in } B; \\ u=0, & \text{on } \partial B, \end{cases}$$

where B is the unit ball in $\mathbb{R}^N$, the radial positive weight $\omega(x)$ is of logarithmic type function, the functional $f(x,u)$ is continuous in $B\times\mathbb{R}$ and has double exponential critical growth, which behaves like $\exp\Big\{e^{\alpha|u|^{\frac{N}{N-1}}}\Big\}$ as $|u|\to\infty$ for some $\alpha>0$. Moreover, $\epsilon>0$, and the radial function h belongs to the dual space of $W^{1,N}_{0,rad}(B)$, $h\neq 0$.

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1 Introduction

In this paper, we deal with the existence and multiplicity of nontrivial solutions for the following nonhomogeneous problem

$$\begin{cases} -\mathrm{div}\Big(\omega(x)|\nabla u(x)|^{N-2}\nabla u(x)\Big)=f(x,u)+\varepsilon h(x), & \text{in } B; \\ u=0, & \text{on } \partial B, \end{cases} \tag{1.1}$$

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where B is the unit ball in $\mathbb{R}^N$, the radial positive weight $\omega(x)$ is of logarithmic type function, the functional $f(x,u)$ is continuous in $B\times\mathbb{R}$ and has double exponential critical growth, which behaves like $\exp\left\{e^{\alpha|u|^{\frac{N}{N-1}}}\right\}$ as $|u|\to\infty$ for some $\alpha>0$. Moreover, $\varepsilon>0$, and the radial function h belongs to the dual space of $W_0^{1,N}(B)$, $h\neq 0$.

Elliptic equations with exponential growth nonlinearities are motivated by the Trudinger-Moser inequality. Let Ω be a bounded domain in $\mathbb{R}^N$, and denote with $W_0^{1,N}(\Omega)$ the standard first order Sobolev space given by

$$W_0^{1,N}(\Omega)=cl\left\{u\in C_0^\infty(\Omega)\;:\;\int_\Omega|\nabla u|^N\mathrm{d}x<\infty\right\},\qquad \|u\|_{W_0^{1,N}(\Omega)}=\left(\int_\Omega|\nabla u|^N\mathrm{d}x\right)^{\frac{1}{N}}.$$

This space is a limiting case for the Sobolev embedding theorem, which yields $W_0^{1,N}(\Omega)\hookrightarrow L^p(\Omega)$ for all $1\le p<\infty$, but one knows by easy examples that $W_0^{1,N}(\Omega)\not\subseteq L^\infty(\Omega)$. Hence, one is led to look for a function $g(s):\mathbb{R}\to\mathbb{R}^+$ with maximal growth such that

$$\sup_{u\in W_0^{1,N}(\Omega),\|u\|_{W_0^{1,N}(\Omega)}\le 1}\int_\Omega g(u)\mathrm{d}x<\infty.$$

It was shown that by Trudinger [1] and Moser [2] that the maximal growth is of exponential type. More precisely,

$$\exp\left(\alpha|u|^{\frac{N}{N-1}}\right)\in L^1(\Omega),\qquad \forall\, u\in W_0^{1,N}(\Omega),\quad \forall\alpha>0,$$

and

$$\sup_{\|u\|_{W_0^{1,N}(\Omega)}\le 1}\int_\Omega\exp\left(\alpha|u|^{\frac{N}{N-1}}\right)\mathrm{d}x\le C(N)\in\mathbb{R},\qquad \text{if } \alpha\le\alpha_N,$$

where $\alpha_N=N\omega_{N-1}^{\frac{1}{N-1}}$ and ω_{N-1} is the $(N-1)$-dimensional surface of the unit sphere.

Recently, the influence of weights on limiting inequalities of Trudinger-Moser type has been studied, for example, see [3–5]. Let $B=B_1(0)$ be the unit ball in $\mathbb{R}^N$, if $\omega\in L^1(\Omega)$ is a non-negative function, we introduce the weighted Sobolev space

$$W_0^{1,N}(\Omega,\omega)=cl\left\{u\in C_0^\infty(\Omega):\;\int_\Omega|\nabla u|^N\omega(x)\mathrm{d}x<\infty\right\}. \tag{1.2}$$

A general embedding theory for such weighted Sobolev spaces has been developed in Kufner [6]. It turns out that for weighted Sobolev spaces of form (1.2) logarithmic weights have a particular significance, since they concern limiting situations of such embeddings. However, to obtain interesting results, one needs to restrict attention to radial functions. So let us consider the subspace of radial functions, i.e.,

$$W_{0,\mathrm{rad}}^{1,N}(B,\omega)=cl\left\{u\in C_{0,\mathrm{rad}}^\infty(B):\;\|u\|:=\int_\Omega|\nabla u|^N\omega(x)\mathrm{d}x<\infty\right\},$$