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On a Nonhomogeneous N-Laplacian Problem with Double Exponential Critical Growth

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Abstract. This paper is devoted to study the existence and multiplicity of nontrivial solutions for the following boundary value problem

$$\begin{cases} -\mathrm{div}\Big(\omega(x)|\nabla u(x)|^{N-2}\nabla u(x)\Big) = f(x,u)+\epsilon h(x), & \text{in } B; \\ u=0, & \text{on } \partial B, \end{cases}$$

where B is the unit ball in $\mathbb{R}^N$, the radial positive weight $\omega(x)$ is of logarithmic type function, the functional $f(x,u)$ is continuous in $B\times\mathbb{R}$ and has double exponential critical growth, which behaves like $\exp\left\{e^{\alpha|u|^{\frac{N}{N-1}}}\right\}$ as $|u|\to\infty$ for some $\alpha>0$. Moreover, $\epsilon>0$, and the radial function h belongs to the dual space of $W^{1,N}_{0,rad}(B)$, $h\neq 0$.

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1 Introduction

In this paper, we deal with the existence and multiplicity of nontrivial solutions for the following nonhomogeneous problem

$$\begin{cases} -\mathrm{div}\Big(\omega(x)|\nabla u(x)|^{N-2}\nabla u(x)\Big) = f(x,u)+\varepsilon h(x), & \text{in } B; \\ u=0, & \text{on } \partial B, \end{cases} \tag{1.1}$$

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where B is the unit ball in $\mathbb{R}^N$, the radial positive weight $\omega(x)$ is of logarithmic type function, the functional $f(x,u)$ is continuous in $B\times\mathbb{R}$ and has double exponential critical growth, which behaves like $\exp\left\{e^{\alpha|u|^{\frac{N}{N-1}}}\right\}$ as $|u|\to\infty$ for some $\alpha>0$. Moreover, $\varepsilon>0$, and the radial function h belongs to the dual space of $W_0^{1,N}(B)$, $h\neq 0$.

Elliptic equations with exponential growth nonlinearities are motivated by the Trudinger-Moser inequality. Let Ω be a bounded domain in $\mathbb{R}^N$, and denote with $W_0^{1,N}(\Omega)$ the standard first order Sobolev space given by

$$W_0^{1,N}(\Omega)=cl\left\{u\in C_0^\infty(\Omega)\;:\;\int_\Omega|\nabla u|^N\mathrm{d}x<\infty\right\},\qquad \|u\|_{W_0^{1,N}(\Omega)}=\left(\int_\Omega|\nabla u|^N\mathrm{d}x\right)^{\frac{1}{N}}.$$

This space is a limiting case for the Sobolev embedding theorem, which yields $W_0^{1,N}(\Omega)\hookrightarrow L^p(\Omega)$ for all $1\le p<\infty$, but one knows by easy examples that $W_0^{1,N}(\Omega)\not\subseteq L^\infty(\Omega)$. Hence, one is led to look for a function $g(s):\mathbb{R}\to\mathbb{R}^+$ with maximal growth such that

$$\sup_{u\in W_0^{1,N}(\Omega),\|u\|_{W_0^{1,N}(\Omega)}\le 1}\int_\Omega g(u)\mathrm{d}x<\infty.$$

It was shown that by Trudinger [1] and Moser [2] that the maximal growth is of exponential type. More precisely,

$$\exp\left(\alpha|u|^{\frac{N}{N-1}}\right)\in L^1(\Omega),\qquad \forall\, u\in W_0^{1,N}(\Omega),\quad \forall\alpha>0,$$

and

$$\sup_{\|u\|_{W_0^{1,N}(\Omega)}\le 1}\int_\Omega\exp\left(\alpha|u|^{\frac{N}{N-1}}\right)\mathrm{d}x\le C(N)\in\mathbb{R},\qquad \text{if } \alpha\le\alpha_N,$$

where $\alpha_N=N\omega_{N-1}^{\frac{1}{N-1}}$ and ω_{N-1} is the $(N-1)$-dimensional surface of the unit sphere.

Recently, the influence of weights on limiting inequalities of Trudinger-Moser type has been studied, for example, see [3–5]. Let $B=B_1(0)$ be the unit ball in $\mathbb{R}^N$, if $\omega\in L^1(\Omega)$ is a non-negative function, we introduce the weighted Sobolev space

$$W_0^{1,N}(\Omega,\omega)=cl\left\{u\in C_0^\infty(\Omega):\;\int_\Omega|\nabla u|^N\omega(x)\mathrm{d}x<\infty\right\}. \tag{1.2}$$

A general embedding theory for such weighted Sobolev spaces has been developed in Kufner [6]. It turns out that for weighted Sobolev spaces of form (1.2) logarithmic weights have a particular significance, since they concern limiting situations of such embeddings. However, to obtain interesting results, one needs to restrict attention to radial functions. So let us consider the subspace of radial functions, i.e.,

$$W_{0,\mathrm{rad}}^{1,N}(B,\omega)=cl\left\{u\in C_{0,\mathrm{rad}}^\infty(B):\;\|u\|:=\int_\Omega|\nabla u|^N\omega(x)\mathrm{d}x<\infty\right\},$$

with the specific weight

$$\omega=\left(\log\frac{e}{|x|}\right)^{N-1}. \tag{1.3}$$

Calanchi and Ruf [5] showed the following double exponential: setting $N'=\frac{N}{N-1}$,

$$\int_B e^{e^{|u|^{N'}}}\mathrm{d}x<\infty, \qquad \forall\, u\in W^{1,N}_{0,\mathrm{rad}}(B,\omega), \tag{1.4}$$

where ω is given by (1.3), and

$$\sup_{u\in W^{1,N}_{0,\mathrm{rad}}(B,\omega),\ \|u\|\le 1}\int_B e^{\beta e^{\omega_{N-1}^{\frac{1}{N-1}}|u|^{N'}}}\mathrm{d}x<\infty \iff \beta\le N. \tag{1.5}$$

The problems of type (1.1) with double exponential growth nonlinearities are motivated from logarithminc weights Trudinger-Moser type inequalities (1.4) and (1.5). If $N=2$, in the semilinear case, the results of Trudinger-Moser type inequalities with logarithmic weights has been obtained in [3, 4], and we refer to [7] for some applications about the existence and multiplicity of solutions for elliptic problems. If $\varepsilon=0$, the existence of solutions for problem (1.1) with double critical exponential nonlinearity at infinity has been studied in [8]. However, as far as we know, there are no results on the existence of solutions for a nonhomogeneous N-Laplacian problem with critical growth of *double exponential type*.

Motivated by the above results, in the present paper, we study the existence and multiplicity of solutions for a nonhomogeneous N-Laplacian problem with double critical exponential nonlinearity. The main purpose of this paper is to establish the result of multiplicity solutions to problem (1.1) when $\varepsilon>0$ small enough by using the Ekeland's variational principle and mountain pass theorem.

In view of inequality (1.5), we say that f has *double exponential subcritical growth* at $+\infty$ if for all $\alpha>0$,

$$\lim_{s\to\infty}\frac{|f(x,s)|}{\exp\left\{e^{\alpha|s|^{\frac{N}{N-1}}}\right\}}=0, \tag{1.6}$$

and f has *double exponential critical growth* at $+\infty$ if there exists $\alpha_0>0$ such that

$$\begin{aligned}
&\lim_{s\to\infty}\frac{|f(x,s)|}{\exp\left\{Ne^{\alpha|s|^{\frac{N}{N-1}}}\right\}}=0, && \forall\,\alpha>\alpha_0;\\
&\lim_{s\to\infty}\frac{|f(x,s)|}{\exp\left\{Ne^{\alpha|s|^{\frac{N}{N-1}}}\right\}}=+\infty, && \forall\alpha\le\alpha_0.
\end{aligned} \tag{1.7}$$

We assume the following conditions on the nonlinearity $f(x,u)$:
(F_1) $f: B\times\mathbb{R}\to\mathbb{R}$ is continuous radial in x, and $f(x,u)=0$ for all $(x,u)\in B\times(-\infty,0]$.
(F_2) There exist constants $R_0, M_0>0$ such that for all $x\in B$ and $u\geq R_0$,

$$F(x,u)\leq M_0 f(x,u).$$

(F_3) There exists $\mu>N$ such that for all $x\in B$ and $u>0$,

$$0<\mu F(x,u)\leq u f(x,u),$$

where $F(x,u)=\int_0^u f(x,t)\mathrm{d}t$.

(F_4) $\limsup_{u\to 0^+}\frac{NF(x,u)}{u^N}<\lambda_1$, uniformly in $x\in B$, where $\lambda_1>0$ is the first eigenvalue associated to the operator $-\mathrm{div}(\omega(x)|\nabla\cdot|^{N-1})$ in $W^{1,N}_{0,\mathrm{rad}}(B,\omega)$.

Since we are only concerned with the nonnegative solution, the condition (F_1) is natural. Our results state as follows.

Theorem 1.1. *Suppose f has critical growth at $+\infty$ with $\alpha_0>0$ given by* (1.7), *and $(F_1)-(F_4)$ hold. Then there exists $\varepsilon_1>0$ such that for each $0<\varepsilon<\varepsilon_1$, problem* (1.1) *has a nontrivial solution with negative energy.*

Theorem 1.2. *Suppose f has critical growth at $+\infty$ with $\alpha_0>0$ given by* (1.7), *and $(F_1)-(F_4)$ hold. Moreover, assume f satisfies the following condition:*

(F_5) *there exists constant γ_0 with $\gamma_0>\frac{1}{\alpha_0^{N-1}e^N}$ such that*

$$\lim_{t\to\infty}\frac{f(x,t)t}{\exp\left\{Ne^{\alpha_0|t|^{\frac{N}{N-1}}}\right\}}\geq\gamma_0,\quad\text{uniformly in } x.$$

Then there exists $\varepsilon_2>0$ such that for each $0<\varepsilon<\varepsilon_2$, problem (1.1) *has at least two nontrivial weak solutions.*

The existence of solutions for critical exponential problems was studied in [9–13] and references therein. Moreover, the existence of solutions for elliptic equations involving critical exponential nonlinearities and a small nonhomogeneous term was considered by many authors, for example, we refer to [14–20]. Here we extend some of these works to consider the nonlinear term has *double exponential critical growth* at infinity given by (1.7).

Since the nonlinearity f has critical growth, the Euler-Lagrange functional does not satisfy the Palais-Smale condition at all level, we will use a logarithmic concentrating sequence (Moser sequence) to show that the functional satisfies the Palais-Smale at a certain level.

The paper is organized as follows: we give some preliminaries results in Section 2. Section 3 is devoted to study the geometry of the Lagrange-Euler function of problem. We give a more precise information about the minimax level obtained by the mountain pass theorem in Section 4. Section 5 is devoted to analyze the compactness of Palais-Smale sequence, and we prove the existence of solutions for problem (1.1) in Section 6.

2 Preliminaries results

Let us consider the space $H:=W^{1,N}_{0,\mathrm{rad}}(B,\omega)$ endowed with the norm

$$\|u\|=\left(\int_B|\nabla u|^N\omega(x)\mathrm{d}x\right)^{\frac{1}{N}} \text{ with } \omega(x)=\left(\log\frac{e}{|x|}\right)^{N-1}.$$

The functional $I_\varepsilon:H\to\mathbb{R}$ is given by

$$I_\varepsilon(u)=\frac{1}{N}\|u\|^N-\int_B F(x,u)\mathrm{d}x-\varepsilon\int_B hu\mathrm{d}x.$$

The functional is of class C^1, since the hypothesis on the growth of f ensures the existence of positive constants c and C such that

$$|f(x,t)|\le C\exp\left\{e^{c|t|^{\frac{N}{N-1}}}\right\}, \qquad \forall x\in B,\ \forall t\in\mathbb{R}. \tag{2.1}$$

A straightforward calculation shows

$$I'_\varepsilon(u)\phi=\int_B|\nabla u|^{N-2}\nabla u\nabla\phi\,\omega(x)\mathrm{d}x-\int_B f(x,u)\phi\mathrm{d}x-\varepsilon\int_B h\phi\mathrm{d}x,$$

for all $\phi\in H$. Hence, a weak solution of (1.1) is a critical point of I_ε.

Definition 2.1. *Let $(X,\|\cdot\|_X)$ be a real Banach space with its dual space $(X^*,\|\cdot\|_{X^*})$ and $I\in C^1(X,\mathbb{R})$. For $c\in\mathbb{R}$, we say that I satisfies the $(PS)_c$ condition if for any sequence $\{u_k\}\subset X$ with $I(u_k)\to c$, $I'(u_k)\to 0$ in X^*, there is a subsequence $\{u_{k_l}\}$ such that $\{u_{k_l}\}$ converges strongly in X.*

We conclude this section with a technical result which we shall use later, whose proof can be seen [8, Lemma 2.1], we give the proof here for the convenience of readers.

Lemma 2.1 (Lions-type Lemma). *Let $\{u_k\}_k\subset H$ be such that $\|u_k\|=1$. If $u_k\rightharpoonup u$ in H and $u\neq 0$, then*

$$\sup_k\int_B e^{Ne^{p\omega_{N-1}^{\frac{1}{N-1}}|u_k|^{\frac{N}{N-1}}}}\mathrm{d}x<+\infty$$

for any $1<p<P$, where

$$P:=\begin{cases}\frac{1}{(1-\|u\|^N)^{\frac{1}{N}}}, & \text{if } \|u\|<1;\\ +\infty, & \text{if } \|u\|=1.\end{cases}$$

Proof. From the Brézis-Lieb Lemma ([21]), it holds that

$$\|u_k-u\|^N=1-\|u\|^N+o_k(1), \tag{2.2}$$

where $o_k(1)\to 0$ as $k\to\infty$. For every $x\in B$, it is not difficult to see that

$$|u_k|^{\frac{N}{N-1}}\le(1+\varepsilon)|u_k-u|^{\frac{N}{N-1}}+C|u|^{\frac{N}{N-1}}$$

for some positive constant C depending only on N and ε, where ε is a small positive number to be chosen later. This together with Young inequality, implies that

$$\begin{aligned}
&\int_B e^{Ne^{p\omega_{N-1}^{\frac{1}{N-1}}|u_k|^{\frac{N}{N-1}}}}\mathrm{d}x\\
\le&\int_B \exp\Big\{Ne^{p\omega_{N-1}^{\frac{1}{N-1}}(1+\varepsilon)|u_k-u|^{\frac{N}{N-1}}}e^{p\omega_{N-1}^{\frac{1}{N-1}}C|u|^{\frac{N}{N-1}}}\Big\}\mathrm{d}x\\
\le&\int_B \exp\Big\{N\Big[\frac{1}{q}e^{qp\omega_{N-1}^{\frac{1}{N-1}}(1+\varepsilon)|u_k-u|^{\frac{N}{N-1}}}+\frac{1}{q'}e^{q'p\omega_{N-1}^{\frac{1}{N-1}}C|u|^{\frac{N}{N-1}}}\Big]\Big\}\mathrm{d}x\\
\le&\frac{1}{q}\int_B \exp\Big\{Ne^{qp\omega_{N-1}^{\frac{1}{N-1}}(1+\varepsilon)|u_k-u|^{\frac{N}{N-1}}}\Big\}\mathrm{d}x+\frac{1}{q'}\int_B\Big\{Ne^{q'p\omega_{N-1}^{\frac{1}{N-1}}C|u|^{\frac{N}{N-1}}}\Big\}\mathrm{d}x,
\end{aligned}\tag{2.3}$$

where $\frac{1}{q}+\frac{1}{q'}=1$. When $p<(1-\|u\|^N)^{-\frac{1}{N-1}}$ and p is fixed, we can choose some $q>1$ and $\varepsilon>0$ such that $qp(1+\varepsilon)<(1-\|u\|^N)^{-\frac{1}{N-1}}$. Then Lemma follows from (1.5), (2.2) and (2.3). □

3 The geometry of the function

In this section, we show that the energy functional J_ε satisfies geometric conditions of the mountain pass theorem. Then, we are going to use the mountain-pass theorem without a compactness condition such as the one of the (PS) type to prove the existence of the solution. This version of the mountain-pass theorem is a consequence of the Ekeland's variational principle.

Lemma 3.1. *Suppose $(F_1)-(F_4)$ hold, f has a critical growth at $+\infty$, then there exists $\varepsilon_1>0$ such that for $0<\varepsilon<\varepsilon_1$, there exists $\rho_\varepsilon>0$ such that*

$$I_\varepsilon(u)>0\ \text{ if }\ \|u\|=\rho_\varepsilon.$$

Furthermore, ρ_ε can be chosen such that $\rho_\varepsilon\to 0$ as $\varepsilon\to 0$.

Proof. From (F_4), there exist $\tau,\delta_0>0$ such that

$$F(x,u)\le\frac{\lambda_1-\tau}{N}|u|^N,\qquad \text{for } |u|\le\delta_0,\ x\in B.\tag{3.1}$$

On the other hand, from (2.1), we have that for $q>N$, there exists a constant $C_1>0$ such that

$$F(x,u)\le C_1|u|^q\exp\Big\{e^{c|u|^{\frac{N}{N-1}}}\Big\},\qquad \forall\ |u|\ge\delta_0,\ \ x\in B.$$

Thus

$$\begin{aligned}I_\varepsilon(u)\geq&\frac{1}{N}\|u\|^N-\frac{\lambda_1-\tau}{N}\int_B|u|^N\mathrm{d}x-C_1\int_B|u|^q\exp\left\{e^{c|u|^{\frac{N}{N-1}}}\right\}\mathrm{d}x-\varepsilon\int_B hu\mathrm{d}x\\ \geq&\frac{\tau}{N\lambda_1}\|u\|^N-C_1\left(\int_B|u|^{\frac{Nq}{N-1}}\mathrm{d}x\right)^{\frac{N-1}{N}}\left(\int_B\exp\left\{Ne^{c|u|^{\frac{N}{N-1}}}\right\}\mathrm{d}x\right)^{\frac{1}{N}}\\ &-\varepsilon\|h\|_*\|u\|.\end{aligned}$$

Now, we choose $\rho>0$ such that $c\rho^{\frac{N}{N-1}}\leq\omega_{N-1}^{\frac{1}{N-1}}$, using (1.5), we find

$$\int_B\exp\left\{Ne^{c|u|^{\frac{N}{N-1}}}\right\}\mathrm{d}x=\int_B\exp\left\{Ne^{c\|u\|^{\frac{N}{N-1}}\left(\frac{|u|}{\|u\|}\right)^{\frac{N}{N-1}}}\right\}\mathrm{d}x\leq C_2,\qquad\forall\, u\in H\text{ with }\|u\|=\rho.$$

Moreover, since

$$\left(\int_B|u|^{\frac{Nq}{N-1}}\mathrm{d}x\right)^{\frac{N-1}{N}}\leq C_3\|u\|^q.$$

We get

$$I_\varepsilon(u)\geq\left[\frac{\tau}{N\lambda_1}\|u\|^{N-1}-C\|u\|^{q-1}-\varepsilon\|h\|_*\right]\|u\| \tag{3.2}$$

for $\rho>0$ satisfying $c\rho^{\frac{N}{N-1}}\leq\omega_{N-1}^{\frac{1}{N-1}}$. Since $q>N$, we may choose $\rho>0$ such that $\frac{\tau}{N}\rho^{N-1}-C\rho^{q-1}>0$ Thus, if ε is sufficiently small then there exists some $\rho_\varepsilon>0$ such that $I_\varepsilon(u)>0$ if $\|u\|=\rho_\varepsilon$ and $\rho_\varepsilon\to0$ as $\varepsilon\to0$. □

Lemma 3.2. *Suppose* (F_1) *and* (F_2) *hold. There exists* $e\in H$ *with* $\|e\|>\rho_\varepsilon$ *such that* $I_\varepsilon(e)<\inf_{\|u\|=\rho_\varepsilon}I_\varepsilon(u)$.

Proof. Let $u_0\in H\cap L^\infty(B)$ such that $\|u_0\|_\infty=1$. From (F_1) and (F_2), there exists a constant $C>0$ such that

$$F(x,u)\geq Ce^{\frac{1}{M_0}|u|},\qquad\forall|u|\geq R_0,\ x\in B.$$

In particular, for $p>N$, there exists C such that

$$F(x,u)\geq C|u|^p-C,\qquad\forall u\in\mathbb{R},\ x\in B.$$

Since $p>N$, for $t>0$, we have

$$I_\varepsilon(tu_0)\leq\frac{t^N}{N}\|u_0\|^N-Ct^p\int_B|u_0|^p\mathrm{d}x+C-t\varepsilon\int_B hu_0\mathrm{d}x\to-\infty$$

as $t\to\infty$. Setting $e=tu_0$ with t sufficiently large, the proof of the lemma follows. □

Lemma 3.3. *Suppose* (F_1) *holds. Then there exist* $\eta>0$ *and* $v\in H$ *with* $\|v\|=1$ *such that* $I_\varepsilon(tv)<0$ *for all* $0<t<\eta$. *In particular,*

$$\inf_{\|u\|\le\eta} I_\varepsilon(u)<0.$$

Proof. Let $v\in H$ be the unique solution of the problem

$$-\operatorname{div}\big(\omega(x)|\nabla v|^{N-2}\nabla v\big)=h \quad \text{in } B, \qquad v=0 \quad \text{on } \partial B.$$

It follows from $h\neq 0$ that

$$\int_B hv\,\mathrm{d}x=\|v\|^N>0.$$

For $t>0$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}I_\varepsilon(tv)=t^{N-1}\|v\|^N-\int_B f(x,tv)v\,\mathrm{d}x-\varepsilon\int_B hv\,\mathrm{d}x.$$

Since $f(x,0)=0$, we have $\frac{\mathrm{d}}{\mathrm{d}t}I_\varepsilon(tv)\big|_{t=0}<0$. By continuity, it follows that there exists $\eta>0$ such that for all $0<t<\eta$,

$$\frac{\mathrm{d}}{\mathrm{d}t}I_\varepsilon(tv)<0.$$

Notice that $I_\varepsilon(0)=0$, we arrive at $I_\varepsilon(tv)<0$ for all $0<t<\eta$. □

4 The minimax level

In order to get a more precise information about the minimax level obtained by the mountain pass theorem, let us consider the function $\varphi_k=\varphi_k(x)$ defined by means of the identity

$$\psi_k(t):=\omega_{N-1}^{\frac{1}{N}}\varphi_k(x), \ \text{ with } |x|=e^{-t}$$

where $\{\psi_k\}_k$ is the Moser-type sequence introduced in [5]. More precisely

$$\psi_k(t)=\begin{cases}\dfrac{\log(1+t)}{\log^{\frac{1}{N}}(1+k)}, & 0\le t\le k;\\ \log^{\frac{N-1}{N}}(1+k), & t\ge k.\end{cases} \tag{4.1}$$

Then

$$\|\varphi_k\|^N=\int_B|\nabla\varphi_k(x)|^N\Big|\log\frac{e}{|x|}\Big|^{N-1}\mathrm{d}x=\int_0^\infty|\psi_k'(t)|^N(1+t)^{N-1}\mathrm{d}t=1,$$

and

$$\int_B\exp\Big\{Ne^{\omega_{N-1}^{\frac{1}{N-1}}|\varphi_k|^{\frac{N}{N-1}}}\Big\}\mathrm{d}x=\omega_{N-1}\int_0^\infty\exp\Big\{Ne^{\psi_k^{\frac{N}{N-1}}}-Nt\Big\}\mathrm{d}t.$$

Lemma 4.1. *We have*

$$\lim_{k\to\infty}\int_0^\infty \exp\left\{Ne^{\psi_k^{\frac{N}{N-1}}}-Nt\right\}\mathrm{d}t=\frac{N+1}{N}e^N. \tag{4.2}$$

Proof. The proof can be seen in [8, Lemma 4.1]. □

Lemma 4.2. *Suppose that (F_1) and (F_5) hold. Then there exists $k\in\mathbb{N}$ such that*

$$\max_{t\geq 0}\left\{\frac{t^N}{N}-\int_B F(x,t\varphi_k(x))\mathrm{d}x\right\}<\frac{1}{N}\left(\frac{\omega_{N-1}^{\frac{1}{N-1}}}{\alpha_0}\right)^{N-1},$$

where $\varphi_k(x)=\omega_{N-1}^{-\frac{1}{N}}\psi_k(t)$ with $|x|=e^{-t}$ and ψ_k is defined in (4.1), *ω_{N-1} is the $(N-1)$-dimensional surface of the unit sphere, and α_0 is given in* (1.7).

Proof. Suppose, by contradiction, that for all k we have

$$\max_{t\geq 0}\left\{\frac{t^N}{N}-\int_B F(x,t\varphi_k(x))\mathrm{d}x\right\}\geq\frac{1}{N}\left(\frac{\omega_{N-1}^{\frac{1}{N-1}}}{\alpha_0}\right)^{N-1}.$$

Then for any $k\geq 1$, there exists $t_k>0$ satisfying

$$\frac{1}{N}\left(\frac{\omega_{N-1}^{\frac{1}{N-1}}}{\alpha_0}\right)^{N-1}\leq\max_{t\geq 0}\left\{\frac{t^N}{N}-\int_B F(x,t\varphi_k(x))\mathrm{d}x\right\}=\frac{t_k^N}{N}-\int_B F(x,t_k\varphi_k(x))\mathrm{d}x.$$

Thus

$$\frac{t_k^N}{N}-\int_B F(x,t_k\varphi_k(x))\mathrm{d}x\geq\frac{1}{N}\left(\frac{\omega_{N-1}^{\frac{1}{N-1}}}{\alpha_0}\right)^{N-1},$$

and using the fact that $F(x,u)\geq 0$, we obtain

$$t_k^N\geq\left(\frac{\omega_{N-1}^{\frac{1}{N-1}}}{\alpha_0}\right)^{N-1}. \tag{4.3}$$

Since at $t=t_k$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{t^N}{N}-\int_B F(x,t\varphi_k(x))\mathrm{d}x\right)=0,$$

it follows that

$$t_k^N=\int_B f(x,t_k\varphi_k)t_k\varphi_k\mathrm{d}x. \tag{4.4}$$

Using hypothesis (F_5), given $\tau>0$, there exists $R_\tau>0$ such that for all $u\geq R_\tau$, we get

$$f(x,t)t\geq(\gamma_0-\tau)\exp\left\{Ne^{\alpha_0|t|^{\frac{N}{N-1}}}\right\},\qquad \forall|t|\geq R_\tau,\text{ uniformly in } x. \tag{4.5}$$

By (4.4) and the definition of φ_k,

$$t_k^N=\int_B f(x,t_k\varphi_k)t_k\varphi_k\mathrm{d}x\geq\omega_{N-1}\int_k^\infty f\left(e^{-s},t_k\frac{\psi_k}{\omega_{N-1}^{1/N}}\right)t_k\frac{\psi_k}{\omega_{N-1}^{1/N}}e^{-Ns}\mathrm{d}s.$$

By the definition of φ_k and (4.3), for $s\geq k$, we have

$$t_k\frac{\psi_k}{\omega_{N-1}^{\frac{1}{N}}}=t_k\left(\frac{\log(1+k)}{\omega_{N-1}^{\frac{1}{N-1}}}\right)^{\frac{N-1}{N}}\geq\left(\frac{\log(1+k)}{\alpha_0}\right)^{\frac{N-1}{N}}.$$

Therefore, for any $k\geq\bar{k}$ with $\bar{k}=\bar{k}(\tau)\geq1$ sufficiently large, from (4.5), we get

$$\begin{aligned}
t_k^N\geq&\omega_{N-1}\int_k^\infty f\left(e^{-s},t_k\frac{\psi_k}{\omega_{N-1}^{\frac{1}{N}}}\right)t_k\frac{\psi_k}{\omega_{N-1}^{\frac{1}{N}}}e^{-Ns}\mathrm{d}s\\
\geq&\omega_{N-1}(\gamma_0-\tau)\int_k^\infty\exp\left\{Ne^{\alpha_0\left|t_k\frac{\psi_k}{\omega_{N-1}^{1/N}}\right|^{\frac{N}{N-1}}}\right\}e^{-Ns}\mathrm{d}s\\
=&\omega_{N-1}(\gamma_0-\tau)\int_k^\infty\exp\left\{Ne^{\alpha_0|t_k|^{\frac{N}{N-1}}\omega_{N-1}^{-\frac{1}{N-1}}\log(1+k)}-Ns\right\}\mathrm{d}s\\
=&\frac{\omega_{N-1}}{N}(\gamma_0-\tau)\exp\left\{Ne^{\alpha_0\omega_{N-1}^{-\frac{1}{N-1}}|t_k|^{\frac{N}{N-1}}\log(1+k)}-Nk\right\}.
\end{aligned}\tag{4.6}$$

So

$$1\geq\frac{\omega_{N-1}}{N}(\gamma_0-\tau)\exp\left\{Ne^{\alpha_0\omega_{N-1}^{-\frac{1}{N-1}}|t_k|^{\frac{N}{N-1}}\log(1+k)}-Nk-N\log t_k\right\},\qquad\forall\, k\geq\bar{k},$$

and thus $\{t_k\}_k$ is bounded. Now, if

$$\lim_{k\to\infty}t_k^N>\left(\frac{\omega_{N-1}^{\frac{1}{N-1}}}{\alpha_0}\right)^{N-1}. \tag{4.7}$$

Then (4.6) would yield a contradiction with the boundedness of $\{t_k\}_k$. Hence (4.7) can not hold, it follows from (4.3) that

$$\lim_{k\to\infty}t_k^N=\left(\frac{\omega_{N-1}^{\frac{1}{N-1}}}{\alpha_0}\right)^{N-1}. \tag{4.8}$$

In order to estimate (4.4) more precisely, we consider the sets

$$A_k=\{x\in B: t_k\varphi_k\geq R_\tau\},\qquad C_k=B\backslash A_k,$$

where $R_\tau>0$ is given by (4.5). By construction,

$$\begin{aligned}t_k^N=&\int_B f(x,t_k\varphi_k)t_k\varphi_k\mathrm{d}x\\ \geq&(\gamma_0-\tau)\int_{A_k}\exp\left\{Ne^{\alpha_0|t_k\varphi_k|^{\frac{N}{N-1}}}\right\}\mathrm{d}x+\int_{C_k}f(x,t_k\varphi_k)t_k\varphi_k\mathrm{d}x\\ \geq&(\gamma_0-\tau)\int_B\exp\left\{Ne^{\alpha_0|t_k\varphi_k|^{\frac{N}{N-1}}}\right\}\mathrm{d}x-(\gamma_0-\tau)\int_{C_k}\exp\left\{Ne^{\alpha_0|t_k\varphi_k|^{\frac{N}{N-1}}}\right\}\mathrm{d}x\\ &+\int_{C_k}f(x,t_k\varphi_k)t_k\varphi_k\mathrm{d}x.\end{aligned}$$

Since $\varphi_k\to 0$ and the characteristic functions $\chi_{C_k}\to 1$ for almost every x in B. Therefore, the Lebesgue's dominated convergence theorem implies

$$\int_{C_k}\exp\left\{Ne^{\alpha_0|t_k\varphi_k|^{\frac{N}{N-1}}}\right\}\mathrm{d}x\to\frac{\omega_{N-1}}{N}e^N\quad\text{and}\quad\int_{C_k}f(x,t_k\varphi_k)t_k\varphi_k\mathrm{d}x\to 0.$$

Then, we have

$$\left(\frac{\omega_{N-1}^{\frac{1}{N-1}}}{\alpha_0}\right)^{N-1}=\lim_{k\to\infty}t_k^N\geq(\gamma_0-\tau)\lim_{k\to\infty}\int_B\exp\left\{Ne^{\alpha_0|t_k|^{\frac{N}{N-1}}|\varphi_k|^{\frac{N}{N-1}}}\right\}\mathrm{d}x-(\gamma_0-\tau)\frac{\omega_{N-1}}{N}e^N.$$

By (4.3) and Lemma 4.1, we have

$$\begin{aligned}&\lim_{k\to\infty}\int_B\exp\left\{Ne^{\alpha_0|t_k|^{\frac{N}{N-1}}|\varphi_k|^{\frac{N}{N-1}}}\right\}\mathrm{d}x\\ =&\omega_{N-1}\lim_{k\to\infty}\int_0^\infty\exp\left\{Ne^{\alpha_0|t_k|^{\frac{N}{N-1}}\omega_{N-1}^{-\frac{1}{N-1}}|\psi_k(t)|^{\frac{N}{N-1}}}-Nt\right\}\mathrm{d}t\\ \geq&\omega_{N-1}\lim_{k\to\infty}\int_0^\infty\exp\left\{Ne^{|\psi_k(t)|^{\frac{N}{N-1}}}-Nt\right\}\mathrm{d}t\\ =&\omega_{N-1}\frac{N+1}{N}e^N.\end{aligned}$$

Thus, we get

$$\left(\frac{\omega_{N-1}^{\frac{1}{N-1}}}{\alpha_0}\right)^{N-1}\geq(\gamma_0-\tau)\omega_{N-1}e^N,$$

which implies that

$$\gamma_0\leq\frac{1}{\alpha_0^{N-1}e^N},$$

a contradiction with (F_5), thus the proof is complete. □

Lemma 4.3. *Suppose that* $(F_1)-(F_5)$ *hold, if* $\varepsilon>0$ *is sufficiently small, then*

$$\max_{t\geq 0} I_\varepsilon(t\varphi_k)=\max_{t\geq 0}\left\{\frac{t^N}{N}-\int_B F(x,t\varphi_k(x))\mathrm{d}x-\varepsilon\int_B th\varphi_k\mathrm{d}x\right\}<\frac{1}{N}\left(\frac{\omega_{N-1}^{\frac{1}{N-1}}}{\alpha_0}\right)^{N-1}.$$

Proof. Since

$$\left|\int_B \varepsilon h\varphi_k\mathrm{d}x\right|\leq\varepsilon\|h\|_*.$$

Then taking ε sufficiently small and using Lemma 4.2, the result follows. □

We can conclude by inequality (3.2) and Lemma 3.3 that

$$-\infty<c_0:=\inf_{\|u\|\leq\eta} I_\varepsilon(u)<0. \tag{4.9}$$

In the next section, we prove that this infimum is achieved and generate a solution. In order to obtain convergence results, we need to improve the estimate of Lemma 4.2.

Lemma 4.4. *Suppose that* $(F_1)-(F_5)$ *hold, then there exist* $\varepsilon_2\in(0,\varepsilon_1]$ *and* $u\in H$ *such that for all* $0<\varepsilon<\varepsilon_2$,

$$I_\varepsilon(tu)<c_0+\frac{1}{N}\left(\frac{\omega_{N-1}^{\frac{1}{N-1}}}{\alpha_0}\right)^{N-1} \qquad \text{for all } t\geq 0.$$

Proof. It is possible to increase the infimum c_0 by reducing ε. By Lemma 3.1, $\rho_\varepsilon\to 0$ as $\varepsilon\to 0$. Consequently, $c_0\to 0$ as $\varepsilon\to 0$. Thus there exists $\varepsilon_2>0$ such that of $0<\varepsilon<\varepsilon_2$, then by Lemma 4.3, we have

$$\max_{t\geq 0} I_\varepsilon(t\varphi_k)<c_0+\frac{1}{N}\left(\frac{\omega_{N-1}^{\frac{1}{N-1}}}{\alpha_0}\right)^{N-1}.$$

Taking $u=\varphi_k$, the result follows. □

5 Palais-Smale sequences

In this section, we are going to prove some properties on the Palais-Smale sequences of I_ε.

Lemma 5.1. *Any Palais-Smale sequence for* I_ε *is bounded and weakly convergent to a weak solution of* (1.1).

Proof. In view of Lemmas 3.1 and 3.2, we can apply the mountain pass theorem to obtain a sequence $\{u_k\}_k \subset H$ such that $I_\varepsilon(u_k) \to c > 0$ and $I'_\varepsilon(u_k) \to 0$ as $k \to \infty$, that is

$$\frac{1}{N}\|u_k\|^N - \int_B F(x,u_k)\mathrm{d}x - \int_B \varepsilon h u_k \mathrm{d}x \to c \quad \text{as } k \to \infty, \tag{5.1}$$
$$|I'_\varepsilon(u_k)v| \le o_k(1)\|v\| \qquad \text{for all } v \in H, \tag{5.2}$$

$$\frac{1}{N}\|u_k\|^N - \int_B F(x,u_k)\mathrm{d}x - \int_B \varepsilon h u_k \mathrm{d}x \to c \quad \text{as } k \to \infty, \tag{5.3}$$
$$\left|I'_\varepsilon(u_k)v\right| \le o_k(1)\|v\| \qquad \text{for all } v \in H, \tag{5.4}$$

where $o_k(1) \to 0$ as $k \to \infty$. Furthermore, by Lemma 4.3, we have that

$$c < \frac{1}{N}\left(\frac{\omega_{N-1}^{\frac{1}{N-1}}}{\alpha_0}\right)^{N-1}.$$

Using (5.3), (5.4) and (F_3), we have

$$\left(\frac{\mu}{N} - 1\right)\|u_k\|^N \le C(1 + \|u_k\|),$$

and hence $\|u_k\|$ is bounded and thus

$$\int_B f(x,u_k)\mathrm{d}x \le C, \qquad \int_B F(x,u_k)\mathrm{d}x \le C.$$

The embedding $H \hookrightarrow L^q(B)$ is compact for all $q \ge N$, by extracting a subsequence, we can assume that $u_k \rightharpoonup u$ weakly in H and $u_k \to u$ for almost all $x \in B$ as $k \to \infty$. Thanks to Lemma 2.1 in [9], we have

$$f(x,u_k) \to f(x,u) \quad \text{in } L^1(B) \text{ as } k \to \infty. \tag{5.5}$$

Since

$$0 < F(x,t) \le M_0 f(x,t) \text{ for all } |t| \ge R_0, \text{ uniformly in } B.$$

We may apply the Lebesgue's dominated convergence theorem to conclude that

$$F(x,u_k) \to F(x,u) \quad \text{in } L^1(B) \text{ as } k \to \infty.$$

Thus by (5.4) passing to the limit, we have

$$\int_B |\nabla u|^{N-2}\nabla v\, \omega(x)\mathrm{d}x - \int_B f(x,u)v\mathrm{d}x - \varepsilon\int_B huv\mathrm{d}x = 0, \qquad \forall\, v \in C_0^\infty(B),$$

and u is a weak solution of (1.1). Moreover, $u \neq 0$ because $c \neq 0$. $\square$

Lemma 5.2. *If $\{u_k\}_k$ is a (PS) sequence for I_ε at any level with*

$$\liminf_{k\to\infty}\|u_k\|^N<\left(\frac{\omega_{N-1}^{\frac{1}{N-1}}}{\alpha_0}\right)^{N-1}, \tag{5.6}$$

then $\{u_k\}_k$ possesses a subsequence which converges strongly to a weak solution u_0 of (1.1).

Proof. Up to a subsequence, we can assume

$$\lim_{k\to\infty}\|u_k\|^N<\left(\frac{\omega_{N-1}^{\frac{1}{N-1}}}{\alpha_0}\right)^{N-1}. \tag{5.7}$$

Thanks to Lemma 5.1, there is a subsequence of $\{u_k\}_k$, denoted by itself, such that $u_k\rightharpoonup u$ weakly in H, where u_0 is a weak solution of (1.1). Let $u_k=u_0+w_k$. It follows that $w_k\rightharpoonup 0$ weakly in H. Hence $w_k\to 0$ strongly in $L^q(B)$ for any $q\geq 1$. The Brezis-Lieb Lemma [21] implies

$$\|u_k\|^N=\|u_0\|^N+\|w_k\|^N+o_k(1). \tag{5.8}$$

Now we are proving the following

$$\lim_{k\to\infty}\int_B f(x,u_k)u_0\mathrm{d}x=\int_B f(x,u_0)u_0\mathrm{d}x. \tag{5.9}$$

In fact, since $u_0\in H$, given $\tau>0$, there exists $\varphi\in C_{0,\mathrm{rad}}^\infty(B)$ such that $\|\varphi-u_0\|<\tau$. We have that

$$\begin{aligned}&\left|\int_B f(x,u_k)u_0\mathrm{d}x-\int_B f(x,u_0)u_0\mathrm{d}x\right|\\ \leq&\left|\int_B f(x,u_k)(u_0-\varphi)\mathrm{d}x\right|+\|\varphi\|_\infty\int_B|f(x,u_k)-f(x,u_0)|\mathrm{d}x\\ &+\left|\int_B f(x,u_0)(u_0-\varphi)\mathrm{d}x\right|.\end{aligned} \tag{5.10}$$

Since $|I_\varepsilon'(u_k)(u_0-\varphi)|\leq\tau_k\|u_0-\varphi\|$ with $\tau_k\to 0$ as $k\to\infty$, we find

$$\begin{aligned}\left|\int_B f(x,u_k)(u_0-\varphi)\mathrm{d}x\right|&\leq\tau_k\|u_0-\varphi\|+\int_B|\nabla u_k|^{N-1}\nabla(u_0-\varphi)\ \omega(x)\mathrm{d}x+\varepsilon\|h\|_*\|u_0-\varphi\|\\ &\leq\tau_k\|u_0-\varphi\|+\|u_k\|\,\|u_0-\varphi\|+\varepsilon\|h\|_*\|u_0-\varphi\|\\ &\leq C\|u_0-\varphi\|\leq C\tau,\end{aligned} \tag{5.11}$$

where C is a positive constant which is independent of k and τ. Similarly, by using the fact $I_\varepsilon'(u_0)(u_0-\varphi)=0$, we have

$$\left|\int_B f(x,u_0)(u_0-\varphi)\mathrm{d}x\right|\leq C\tau. \tag{5.12}$$

Thus, from (5.10)-(5.12) and (5.5), we obtain that (5.9) holds. From (5.8) and (5.9), we can write

$$\begin{aligned}I'_\varepsilon(u_k)u_k=&\|u_k\|^N-\int_B f(x,u_k)u_k\mathrm{d}x-\varepsilon\int_B hu_k\mathrm{d}x\\=&\|u_0\|^N-\int_B f(x,u_0)u_0\mathrm{d}x-\varepsilon\int_B hu_0\mathrm{d}x+\|w_k\|^N-\int_B f(x,u_k)(u_k-u_0)\mathrm{d}x\\&-\left[\int_B f(x,u_k)u_0\mathrm{d}x-\int_B f(x,u_0)u_0\mathrm{d}x\right]+o_k(1)\\=&I'_\varepsilon(u_0)u_0+\|w_k\|^N-\int_B f(x,u_k)w_k\mathrm{d}x+o_k(1),\end{aligned}$$

that is

$$\|w_k\|^N=\int_B f(x,u_k)w_k\mathrm{d}x+o_k(1). \tag{5.13}$$

Since f has a critical growth, for every $\tau>0$ and $q>1$, there exist $R_\tau>0$ and $C_{\tau,q}>0$ such that

$$|f(x,t)|^q\le C_{\tau,q}\exp\left\{Ne^{\alpha_0(1+\tau)|t|^{\frac{N}{N-1}}}\right\},\qquad \forall\ |t|\ge R_\tau,\text{ uniformly in } x.$$

Therefore,

$$\begin{aligned}\int_B|f(x,u_k)|^q\mathrm{d}x=&\int_{\{|u_k|\le R_\tau\}}|f(x,u_k)|^q\mathrm{d}x+\int_{\{|u_k|\ge R_\tau\}}|f(x,u_k)|^q\mathrm{d}x\\\le&\pi\max_{B\times[-R_\tau,R_\tau]}|f(x,s)|^q+C_{\tau,q}\int_B\exp\left\{Ne^{\alpha_0(1+\tau)|u_k|^{\frac{N}{N-1}}}\right\}\mathrm{d}x.\end{aligned}$$

From (5.7), there exists k_δ such that

$$\|u_k\|^N\le(1-\delta)^{N-1}\left(\frac{\omega_{N-1}^{\frac{1}{N-1}}}{\alpha_0}\right)^{N-1}\qquad\text{for all } k\ge k_\delta,$$

that is

$$\alpha_0(1+\tau)\|u_k\|^{\frac{N}{N-1}}\le(1+\tau)(1-\delta)\omega_{N-1}^{\frac{1}{N-1}},$$

for all $k\ge k_\delta$. Now, choosing $\tau>0$ sufficiently small, such that

$$\alpha_0(1+\tau)\|u_k\|^{\frac{N}{N-1}}\le\omega_{N-1}^{\frac{1}{N-1}}.$$

Then

$$\int_B\exp\left\{Ne^{\alpha_0(1+\tau)|u_k|^{\frac{N}{N-1}}}\right\}\mathrm{d}x=\int_B\exp\left\{Ne^{\alpha_0(1+\tau)\|u_k\|^{\frac{N}{N-1}}\left(\frac{|u_k|}{\|u_k\|}\right)^{\frac{N}{N-1}}}\right\}\mathrm{d}x$$

$$\leq \int_B \exp\left\{ N e^{\omega_{N-1}^{\frac{1}{N-1}} \left(\frac{|u_k|}{\|u_k\|}\right)^{\frac{N}{N-1}}} \right\} \mathrm{d}x,$$

which is uniformly bounded in view of the weight Trudinger-Moser inequality. Thus

$$\Big| \int_B f(x,u_k) w_k \mathrm{d}x \Big| \leq \left(\int_B |f(x,u_k)|^q \mathrm{d}x \right)^{\frac{1}{q}} \|w_k\|^{q'} \leq C \|w_k\|^{q'} \to 0, \tag{5.14}$$

as $k \to \infty$, where q' is the the conjugate exponent of q. It follows from (5.13) and (5.14) that u_k converges to u_0 strongly in H. □

6 Existence results

6.1 Proof of Theorem 1.1

The proof of the existence of the first solution to (1.1) follows by a minimization argument and the Ekeland's variational principle.

Proposition 6.1. *For each* ε *with* $0 < \varepsilon < \varepsilon_1$*, problem* (1.1) *has a minimum type solution* u_0 *with* $I_\varepsilon(u_0) = c_0 < 0$*, where* c_0 *is defined in* (4.9).

Proof. Let ρ_ε be as in Lemma 3.1. We can choose $\varepsilon_1 > 0$ sufficiently small such that

$$\rho_\varepsilon < \left(\frac{\omega_{N-1}^{\frac{1}{N-1}}}{\alpha_0} \right)^{\frac{N-1}{N}}.$$

Since $\bar{B}_{\rho_\varepsilon}$ is a complete metric space with the metric given by the norm of H, convex and the functional I_ε is of class C^1 and bounded below on $\bar{B}_{\rho_\varepsilon}$, by the Ekeland's variational principle, there exists a sequence $\{u_k\}_k$ in $\bar{B}_{\rho_\varepsilon}$ such that

$$I_\varepsilon(u_k) \to c_0 = \inf_{\|u\| \leq \rho_\varepsilon} I_\varepsilon(u), \quad \text{and} \quad \|I'_\varepsilon(u_k)\| \to 0,$$

as $k \to \infty$. Observing that

$$\|u_k\| \leq \rho_\varepsilon < \left(\frac{\omega_{N-1}^{\frac{1}{N-1}}}{\alpha_0} \right)^{\frac{N-1}{N}}.$$

By Lemma 5.2, it follows that there exists a subsequence of $\{u_k\}$ which converges to a solution u_0 of (1.1). Therefore, $I_\varepsilon(u_0) = c_0 < 0$. □

6.2 Proof of Theorem 1.2

The proof of the existence of the second solution to (1.1) by the mountain pass theorem.

Proposition 6.2. *Suppose that* $(F_1)-(F_5)$ *hold, if* $\varepsilon<\varepsilon_1$*, the problem* (1.1) *has a solution* u_M *via mountain pass theorem.*

Proof. From Lemmas 3.1 and 3.2, J_ε satisfies the hypotheses of the mountain pass theorem except possibly for the Palais-Smale condition. Thus, using the mountain pass theorem without the Palais-Smale condition (see [22]), there exists a sequence $\{u_k\}$ in H satisfying

$$I_\varepsilon(u_k)\to c_m>0 \quad \text{and} \quad \|I'_\varepsilon(u_k)\|\to 0,$$

where c_m is the mountain pass level of I_ε. Now, by Lemma 5.1, the sequence $\{u_k\}$ converges weakly to a solution u_M of (1.1). □

Remark 6.1. By Lemma 4.3, we have

$$0<c_m<c_0+\frac{1}{N}\left(\frac{\omega_{N-1}^{\frac{1}{N-1}}}{\alpha_0}\right)^{N-1}.$$

Proposition 6.3. *If* $\varepsilon_2>0$ *is enough small, then the solutions of* (1.1) *obtained in Propositions 6.1 and 6.2 are distinct.*

Proof. By Propositions 6.1 and 6.2, there exist sequence $\{u_k\}$, $\{v_k\}$ in H such that

$$\begin{aligned}
&u_k\to u_0 \quad \text{and} \quad v_k\rightharpoonup u_M,\\
&I_\varepsilon(u_k)\to c_0<0 \quad \text{and} \quad I_\varepsilon(v_k)\to c_m>0,\\
&I'_\varepsilon(u_k)u_k\to 0 \quad \text{and} \quad I'_\varepsilon(v_k)v_k\to 0.
\end{aligned}$$

Suppose by contradiction that $u_0=u_M$. As the proof in Lemma 5.1, applying Lemma 2.1 in [9], we have

$$f(x,v_k)\to f(x,u_0) \quad \text{in } L^1(B) \text{ as } k\to\infty.$$

Since

$$0<F(x,t)\le M_0 f(x,t) \qquad \text{for all } |t|\ge R_0, \text{ uniformly in } B.$$

We may apply the Lebesgue's dominated convergence Theorem to conclude that

$$F(x,v_k)\to F(x,u_0) \quad \text{in } L^1(B) \text{ as } k\to\infty. \tag{6.1}$$

Therefore,

$$\lim_{k\to\infty}\|v_k\|^N=Nc_m+N\int_B F(x,u_0)\mathrm{d}x+N\varepsilon\int_B hu_0\mathrm{d}x.$$

Setting
$$z_k=\frac{v_k}{\|v_k\|}\quad\text{and}\quad z_0=\frac{u_0}{\lim\limits_{k\to\infty}\|v_k\|},$$
we have that $\|z_k\|=1$ for all k and $z_k\rightharpoonup z_0$ in H, and $\|z_0\|\le 1$.

Now, the following two possibilities would have to occur.

Case 1. $\|z_0\|=1$. In this case, we have $v_k\to u_0$ in H. Thus, we can find $g\in H$, such that $|v_k(x)|\le g(x)$ a.e. in B. It follows from (1.7) and Lebesgue's dominated convergence theorem that
$$\int_B f(x,v_k)v_k\mathrm{d}x\to\int_B f(x,u_0)u_0\mathrm{d}x.$$
Similarly, we have
$$\int_B f(x,u_k)u_k\mathrm{d}x\to\int_B f(x,u_0)u_0\mathrm{d}x.$$
Then, from $I'_\varepsilon(u_k)u_k\to 0$ and $I'_\varepsilon(v_k)v_k\to 0$, we find
$$\lim_{k\to\infty}\|v_k\|=\lim_{k\to\infty}\|u_k\|=\|u_0\|.$$
Noting that $v_k\rightharpoonup u_0$ in H, and using (6.1), we get
$$\lim_{k\to\infty}I_\varepsilon(v_k)=c_m=I_\varepsilon(u_0)=c_0,$$
which is a contradiction.

Case 2. $\|z_0\|<1$. Using Remark 6.1, we have
$$c_m-I_\varepsilon(u_0)<\frac{1}{N}\left(\frac{\omega_{N-1}^{\frac{1}{N-1}}}{\alpha_0}\right)^{N-1},$$
which implies that
$$\alpha_0<\frac{\omega_{N-1}^{\frac{1}{N-1}}}{[N(c_m-I_\varepsilon(u_0))]^{\frac{1}{N-1}}}.$$
Choosing $\tau>0$ sufficiently close to 0 and setting
$$L(\varphi)=c_m+\int_B F(x,\varphi)\mathrm{d}x+\varepsilon\int_B h\varphi\mathrm{d}x.$$
We obtain for some $\delta>0$ that
$$(1+\tau)\alpha_0\|v_k\|^{\frac{N}{N-1}}\le\frac{\omega_{N-1}^{\frac{1}{N-1}}\|v_k\|^{\frac{N}{N-1}}}{[N(c_m-I_\varepsilon(u_0))]^{\frac{1}{N-1}}}-\delta=\frac{\omega_{N-1}^{\frac{1}{N-1}}(NL(v_k))^{\frac{1}{N-1}}+o_k(1)}{[N(c_m-I_\varepsilon(u_0))]^{\frac{1}{N-1}}}-\delta.$$

Notice that

$$L(v_k) = c_m + \int_B F(x,u_0)\mathrm{d}x + \varepsilon \int_B h u_0 \mathrm{d}x + o_k(1),$$

and

$$\left(c_m + \int_B F(x,u_0)\mathrm{d}x + \varepsilon \int_B h u_0 \mathrm{d}x\right)(1 - \|z_0\|^N) \le c_m - I_\varepsilon(u_0),$$

where we have used the definition of z_0 and

$$\int_B F(x,u_0)\mathrm{d}x + \varepsilon \int_B h(x) u_0 \mathrm{d}x = -I_\varepsilon(u_0) + \frac{1}{N}\|u_0\|^N.$$

Thus, for $k>0$ sufficiently large, we get

$$(1+\tau)\alpha_0 \|v_k\|^{\frac{N}{N-1}} \le \frac{\omega_{N-1}^{\frac{1}{N-1}}}{(1-\|z_0\|^N)^{\frac{1}{N-1}}} - \delta < \frac{\omega_{N-1}^{\frac{1}{N-1}}}{(1-\|z_0\|^N)^{\frac{1}{N-1}}}. \tag{6.2}$$

Since f has critical growth, for every $\tau > 0$ and $q > 1$, there exist $R_\tau > 0$ and $C_{\tau,q} > 0$ such that

$$|f(x,t)|^q \le C_{\tau,q} \exp\left\{ N e^{\alpha_0(1+\tau)|t|^{\frac{N}{N-1}}} \right\}, \qquad \forall\ |t| \ge R_\tau, \text{ uniformly in } x.$$

Therefore,

$$\begin{aligned}\int_B |f(x,v_k)|^q \mathrm{d}x &= \int_{\{|u_k| \le R_\tau\}} |f(x,v_k)|^q \mathrm{d}x + \int_{\{|v_k| \ge R_\tau\}} |f(x,v_k)|^q \mathrm{d}x \\ &\le \pi \max_{B \times [-R_\tau, R_\tau]} |f(x,s)|^q + C_{\tau,q} \int_B \exp\left\{ N e^{\alpha_0(1+\tau)|v_k|^{\frac{N}{N-1}}} \right\} \mathrm{d}x.\end{aligned}$$

Using Lemma 2.1 and (6.2), we find

$$\int_B \exp\left\{ N e^{\alpha_0(1+\tau)|v_k|^{\frac{N}{N-1}}} \right\} \mathrm{d}x = \int_B \exp\left\{ N e^{(1+\tau)\alpha_0 \|v_k\|^{\frac{N}{N-1}} (\frac{v_k}{\|v_k\|})^{\frac{N}{N-1}}} \right\} \mathrm{d}x \le C. \tag{6.3}$$

Thus, we get

$$\int_B |f(x,v_k)|^q \mathrm{d}x \le C.$$

By the Hölder inequality,

$$\int_B f(x,v_k)(v_k - u_0)\mathrm{d}x \le \left(\int_B |f(x,v_k)|^q \mathrm{d}x\right)^{\frac{1}{q}} \|v_k - u_0\|_{q'} \to 0,$$

as $k \to \infty$, where $q' = \frac{q}{q-1}$. From this convergence and by $I'_\varepsilon(v_k)(v_k - u_0) \to 0$, we have

$$\int_B \omega(x) |\nabla v_k|^{N-2} \nabla v_k \nabla (v_k - u_0) \mathrm{d}x \to 0 \quad \text{as } k \to \infty.$$

Moreover, since $v_k \rightharpoonup u_0$ in H, we have

$$\int_B \omega(x)|\nabla u_0|^{N-2}\nabla u_0 \nabla(v_k - u_0)\mathrm{d}x \to 0 \quad \text{as } k\to\infty.$$

Using the elementary inequality that (see [23]), for $a,b\in\mathbb{R}^N$,

$$\left(|a|^{N-2}a-|b|^{N-2}b\right)\cdot(a-b) \geq \begin{cases} |a-b|^2, & N=2; \\ \frac{1}{2^{N-1}}|a-b|^N, & N\geq 3, \end{cases}$$

we get

$$\begin{aligned}&\int_B |\nabla v_k - \nabla u_0|^N \,\omega(x)\mathrm{d}x \\ \leq & C\int_B (|\nabla v_k|^{N-2}\nabla v_k - |\nabla u_0|^{N-2}\nabla u_0)(\nabla v_k - \nabla u_0)\,\omega(x)\mathrm{d}x.\end{aligned}$$

This implies that

$$\|v_k - u_0\| \to 0 \quad \text{as } k\to\infty,$$

i.e., $v_k \to u_0$ in H. Thus $I_\varepsilon(v_k)\to I_\varepsilon(u_0)=c_0$, which is a contradiction. Therefore $u_0\neq u_M$ and the proof is complete. □

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