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Incompressible Limit of Nonisentropic Ideal Magnetohydrodynamic Flows with General Initial Data

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Abstract. The incompressible limit of nonisentropic ideal magnetohydrodynamic equations with general initial data in the whole space $\mathbb{R}^3$ is proved in this paper. The uniform estimates of solutions with respect to the Mach number are obtained by using energy estimate. Strong convergence results of the smooth solutions are established by using Strichartz's estimates in the whole space.

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1 Introduction

1.1 The model

We write the three-demensional nonisentropic compressible magnetohydrodynamic equations in $\mathbb{R}^3$ in the following form:

$$\begin{aligned}&\partial_t\rho+\nabla\cdot(\rho u)=0,\\&\rho(\partial_t u+u\cdot\nabla u)+\nabla p+H\times(\nabla\times H)=0,\\&\partial_t H-\nabla\times(u\times H)=0,\qquad \nabla\cdot H=0,\\&\partial_t S+u\cdot\nabla S=0,\end{aligned}\tag{1.1}$$

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where $\rho>0$ is the density of the fliud, $u=(u_1,u_2,u_3)^{\rm T}$ is the velocity, $H=(H_1,H_2,H_3)^{\rm T}$ is the magnetic field, and S is the entropy of the fluid. $p=p(\rho,S)>0$ is the pressure, which is a smooth function of the density and the entropy.

We begin to choose the entropy S and the pressure p as independent thermodynamic variables and let the density ρ be a well-defined function $\rho=\rho(/;;)$, where $\rho(\cdot,\cdot)$ satisfies $\rho>0$ and $\frac{\partial\rho}{\partial p}>0$. Then we rewrite the Eqs. (1.1) in an appropriate nondimensional form

$$
\begin{aligned}
&a(\partial_t p+u\cdot\nabla p)+\nabla\cdot u=0,\\
&\rho(\partial_t u+u\cdot\nabla u)+\frac{1}{\varepsilon^2}\nabla p+H\times(\nabla\times H)=0,\\
&\partial_t H+u\cdot\nabla H-H\cdot\nabla u+H\nabla\cdot u=0,\quad \nabla\cdot H=0,\\
&\partial_t S+u\cdot\nabla S=0,
\end{aligned}
\tag{1.2}
$$

where $a(p,S)=\frac{1}{\rho}\frac{\partial\rho}{\partial p}$ and $\varepsilon>0$ denotes the scaled Mach number for the entity of the slightly compressible fluid. Next, we introduce the following scalings,

$$
p=1+\varepsilon r,\qquad S=1+\varepsilon\Theta, \tag{1.3}
$$

and rewrite the system (1.2) as

$$
\begin{aligned}
&a(\partial_t r^\varepsilon+u^\varepsilon\cdot\nabla r^\varepsilon)+\frac{1}{\varepsilon}\nabla\cdot u^\varepsilon=0,\\
&\rho(\partial_t u^\varepsilon+u^\varepsilon\cdot\nabla u^\varepsilon)+\frac{1}{\varepsilon}\nabla r^\varepsilon+H^\varepsilon\times(\nabla\times H^\varepsilon)=0,\\
&\partial_t H^\varepsilon+u^\varepsilon\cdot\nabla H^\varepsilon-H^\varepsilon\cdot\nabla u^\varepsilon+H^\varepsilon\nabla\cdot u^\varepsilon=0,\quad \nabla\cdot H^\varepsilon=0,\\
&\partial_t\Theta^\varepsilon+u^\varepsilon\cdot\nabla\Theta^\varepsilon=0.
\end{aligned}
\tag{1.4}
$$

Here we notice that a and ρ are dependent on both $\varepsilon r^\varepsilon$ and $\varepsilon\Theta^\varepsilon$.

Setting $U^\varepsilon=(r^\varepsilon,u^\varepsilon,H^\varepsilon,\Theta^\varepsilon)$, we can rewrite the system (1.4) into the following compact symmetric form

$$
A_0\partial_t U^\varepsilon+\sum_{j=1}^{3}\left(A_j+\frac{1}{\varepsilon}C_j\right)\partial_j U^\varepsilon=0, \tag{1.5}
$$

where

$$
A_0=\begin{pmatrix} a & 0 & 0\\ 0 & \rho I_3 & 0\\ 0 & 0 & I_4\end{pmatrix},
$$