JOURNAL OF PARTIAL DIFFERENTIAL EQUATIONS
J. Part. Diff. Eq., Vol. **38**, No. 1, pp. 101-114

doi: 10.4208/jpde.v38.n1.6
March 2025

Incompressible Limit of Nonisentropic Ideal Magnetohydrodynamic Flows with General Initial Data

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Received 24 November 2023; Accepted 27 March 2024

Abstract. The incompressible limit of nonisentropic ideal magnetohydrodynamic equations with general initial data in the whole space $\mathbb{R}^3$ is proved in this paper. The uniform estimates of solutions with respect to the Mach number are obtained by using energy estimate. Strong convergence results of the smooth solutions are established by using Strichartz's estimates in the whole space.

AMS Subject Classifications: 76W05, 35B40, 76M45

Chinese Library Classifications: O175.29, O361.3

Key Words: Incompressible limit; general initial data; nonisentropic MHD equations.

1 Introduction

1.1 The model

We write the three-demensional nonisentropic compressible magnetohydrodynamic equations in $\mathbb{R}^3$ in the following form:

$$\begin{aligned}
&\partial_t\rho+\nabla\cdot(\rho u)=0,\\
&\rho(\partial_t u+u\cdot\nabla u)+\nabla p+H\times(\nabla\times H)=0,\\
&\partial_t H-\nabla\times(u\times H)=0, \qquad \nabla\cdot H=0,\\
&\partial_t S+u\cdot\nabla S=0,
\end{aligned} \tag{1.1}$$

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where $\rho>0$ is the density of the fliud, $u=(u_1,u_2,u_3)^{\mathrm{T}}$ is the velocity, $H=(H_1,H_2,H_3)^{\mathrm{T}}$ is the magnetic field, and S is the entropy of the fluid. $p=p(\rho,S)>0$ is the pressure, which is a smooth function of the density and the entropy.

We begin to choose the entropy S and the pressure p as independent thermodynamic variables and let the density ρ be a well-defined function $\rho=\rho(/;;)$, where $\rho(\cdot,\cdot)$ satisfies $\rho>0$ and $\frac{\partial\rho}{\partial p}>0$. Then we rewrite the Eqs. (1.1) in an appropriate nondimensional form

$$\begin{aligned}
&a(\partial_t p+u\cdot\nabla p)+\nabla\cdot u=0,\\
&\rho(\partial_t u+u\cdot\nabla u)+\frac{1}{\varepsilon^2}\nabla p+H\times(\nabla\times H)=0,\\
&\partial_t H+u\cdot\nabla H-H\cdot\nabla u+H\nabla\cdot u=0,\quad \nabla\cdot H=0,\\
&\partial_t S+u\cdot\nabla S=0,
\end{aligned}\tag{1.2}$$

where $a(p,S)=\frac{1}{\rho}\frac{\partial\rho}{\partial p}$ and $\varepsilon>0$ denotes the scaled Mach number for the entity of the slightly compressible fluid. Next, we introduce the following scalings,

$$p=1+\varepsilon r,\qquad S=1+\varepsilon\Theta,\tag{1.3}$$

and rewrite the system (1.2) as

$$\begin{aligned}
&a(\partial_t r^\varepsilon+u^\varepsilon\cdot\nabla r^\varepsilon)+\frac{1}{\varepsilon}\nabla\cdot u^\varepsilon=0,\\
&\rho(\partial_t u^\varepsilon+u^\varepsilon\cdot\nabla u^\varepsilon)+\frac{1}{\varepsilon}\nabla r^\varepsilon+H^\varepsilon\times(\nabla\times H^\varepsilon)=0,\\
&\partial_t H^\varepsilon+u^\varepsilon\cdot\nabla H^\varepsilon-H^\varepsilon\cdot\nabla u^\varepsilon+H^\varepsilon\nabla\cdot u^\varepsilon=0,\quad \nabla\cdot H^\varepsilon=0,\\
&\partial_t\Theta^\varepsilon+u^\varepsilon\cdot\nabla\Theta^\varepsilon=0.
\end{aligned}\tag{1.4}$$

Here we notice that a and ρ are dependent on both $\varepsilon r^\varepsilon$ and $\varepsilon\Theta^\varepsilon$.

Setting $U^\varepsilon=(r^\varepsilon,u^\varepsilon,H^\varepsilon,\Theta^\varepsilon)$, we can rewrite the system (1.4) into the following compact symmetric form

$$A_0\partial_t U^\varepsilon+\sum_{j=1}^{3}\left(A_j+\frac{1}{\varepsilon}C_j\right)\partial_j U^\varepsilon=0,\tag{1.5}$$

where

$$A_0=\begin{pmatrix} a & 0 & 0\\ 0 & \rho I_3 & 0\\ 0 & 0 & I_4\end{pmatrix},$$

$$A_1=\begin{pmatrix} au_1^\varepsilon & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & \rho u_1^\varepsilon & 0 & 0 & 0 & H_2^\varepsilon & H_3^\varepsilon & 0\\ 0 & 0 & \rho u_1^\varepsilon & 0 & 0 & -H_1^\varepsilon & 0 & 0\\ 0 & 0 & 0 & \rho u_1^\varepsilon & 0 & 0 & -H_1^\varepsilon & 0\\ 0 & 0 & 0 & 0 & u_1^\varepsilon & 0 & 0 & 0\\ 0 & H_2^\varepsilon & -H_1^\varepsilon & 0 & 0 & u_1^\varepsilon & 0 & 0\\ 0 & H_3^\varepsilon & 0 & -H_1^\varepsilon & 0 & 0 & u_1^\varepsilon & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & u_1^\varepsilon \end{pmatrix},$$

$$A_2=\begin{pmatrix} au_2^\varepsilon & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & \rho u_2^\varepsilon & 0 & 0 & -H_2^\varepsilon & 0 & 0 & 0\\ 0 & 0 & \rho u_2^\varepsilon & 0 & H_1^\varepsilon & 0 & H_3^\varepsilon & 0\\ 0 & 0 & 0 & \rho u_2^\varepsilon & 0 & 0 & -H_2^\varepsilon & 0\\ 0 & -H_2^\varepsilon & H_1^\varepsilon & 0 & u_2^\varepsilon & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & u_2^\varepsilon & 0 & 0\\ 0 & 0 & H_3^\varepsilon & -H_2^\varepsilon & 0 & 0 & u_2^\varepsilon & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & u_2^\varepsilon \end{pmatrix},$$

$$A_3=\begin{pmatrix} au_3^\varepsilon & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & \rho u_3^\varepsilon & 0 & 0 & -H_3^\varepsilon & 0 & 0 & 0\\ 0 & 0 & \rho u_3^\varepsilon & 0 & 0 & -H_3^\varepsilon & 0 & 0\\ 0 & 0 & 0 & \rho u_3^\varepsilon & H_1^\varepsilon & H_2^\varepsilon & 0 & 0\\ 0 & -H_3^\varepsilon & 0 & H_1^\varepsilon & u_3^\varepsilon & 0 & 0 & 0\\ 0 & 0 & -H_3^\varepsilon & H_2^\varepsilon & 0 & u_3^\varepsilon & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & u_3^\varepsilon & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & u_3^\varepsilon \end{pmatrix},$$

$$C_j=\begin{pmatrix} 0 & e_j^{\mathrm{T}} & 0\\ e_j & 0 & 0\\ 0 & 0 & 0 \end{pmatrix},\qquad e_j=(\delta_{1j},\delta_{2j},\delta_{3_j})^{\mathrm{T}},\qquad \delta_{ij}=\begin{cases}1, & \text{if } i=j,\\ 0, & \text{if } i\neq j.\end{cases}$$

We notice that the coefficient matrices have the special structures

(1) $A_0=A_0\big(\varepsilon U^\varepsilon\big)$.

(2) For $j=1,2,3$,

$$A_j=A_j\big(U^\varepsilon,\varepsilon U^\varepsilon\big).$$

(3) For $j=1,2,3$,

$$A_0=(A_0)^{\mathrm{T}},\quad A_j=(A_j)^{\mathrm{T}},\quad \text{and}\quad C_j=(C_j)^{\mathrm{T}}.$$

We will consider the incompressible limit of the ideal nonisentropic MHD Eq. (1.5) with the general initial data

$$(r^\varepsilon,u^\varepsilon,H^\varepsilon,\Theta^\varepsilon)|_{t=0}=(r_0^\varepsilon,u_0^\varepsilon,H_0^\varepsilon,\Theta_0^\varepsilon)\quad \text{in } \mathbb{R}^3. \tag{1.6}$$

Assuming that as $\varepsilon\to 0$, the solution $(r^\varepsilon,u^\varepsilon,H^\varepsilon,\Theta^\varepsilon)$ converges to a limit function $U^0=(r^0,u^0,H^0,\Theta^0)$ in some sense, we expect that the limit function satisfies the following incompressible system:

$$\begin{aligned}&\overline{\rho}(\partial_t u^0+u^0\cdot\nabla u^0)+\nabla\pi-H^0\cdot\nabla H^0=0,\\&\partial_t H^0+u^0\cdot\nabla H^0-H^0\cdot\nabla u^0=0,\\&\nabla\cdot u^0=0,\quad \nabla\cdot H^0=0,\end{aligned}\tag{1.7}$$

for some function π, where $\overline{\rho}=\rho(1,1)$.

1.2 Previous results

There are plenty of works on the low Mach number limit to MHD equations in different settings. We shall just mention a few of them.

For the isentropic MHD equations, Klainerman-Majda [1] first studied the incompressible limit of the ideal compressible MHD equations with well-prepared initial data in the spatially periodic case. By the convergence-stability principle, Li [2] obtained the combined incompressible and inviscid limit of the compressible viscous MHD equations with well-prepared initial data. The incompressible limit of the compressible viscous MHD equations with general initial data was studied in [3–5]. For more extended results, we can refer to [4–7].

For the non-isentropic equations, Jiang et al. [8] carried out the work on the incompressible limit of the full compressible MHD equations with well-prepared initial data, where the effect of small temperature variation is taken into consideration, and Cui [9] and Ou [10] also created progress in some extensions on bounded domain. Shortly afterwards, cooperated with Jiang et al. investigated in [11] the low Mach number limit of the full compressible MHD equations with general initial data in the whole space, when the heat conductivity and large temperature variations are present. For the ideal compressible MHD equations, Jiang et al. [12] studied the convergence of solutions with general initial data in the whole space, where they added some additional restrictions to obtain the uniform estimates of solutions. Recently, Li-Zhang [13] studied the incompressible limit of nonisentropic ideal MHD equations with well-prepared initial data in both the whole space and the torus in case of removing additional restrictions mentioned in [12]. By using the original ideas of Schochet [14], Meng-Wang [15] condsidered the incompressible limit of nonisentropic ideal MHD equations with general initial data in the periodic domain, where the effect of small temperature variation is taken into consideration.

1.3 Notations

In this part, we give some notations used throughout this paper. The symbols C or K denote generic positive constants independent of ε, and $f(\cdot)$ and $F(\cdot)$ denote the continuous nondecreasing functions on $[0,\infty)$, which may vary from line to line. $L^p(\mathbb{R}^3)$ $(1\le p<\infty)$ denotes the space of measurable functions whose p-powers are integrable with the norm $\|\cdot\|_{L^p}$, and $L^\infty(\mathbb{R}^3)$ is the space of bounded measurable functions with the norm $\|\cdot\|_{L^\infty}$. We also denote $\|\cdot\|_{L^2}$ by $\|\cdot\|$. We denote by $\langle\cdot,\cdot\rangle$ the standard inner product in $L^2(\mathbb{R}^3)$ with the norm $\|u\|^2=\langle u,u\rangle$, and by H^k the usual Sobolev space $W^{k,2}(\mathbb{R}^3)$ with the norm $\|\cdot\|_k$. The notation $\|(A_1,\cdots,A_k)\|$ means the summation of $\|A_i\|$ $(i=1,\cdots,k)$, and it also applies to other norms. For a multi-index $\alpha=(\alpha_1,\alpha_2,\alpha_3)$, we define $D^\alpha=\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_{x_3}^{\alpha_3}$ and $|\alpha|=|\alpha_1|+|\alpha_2|+|\alpha_3|$. We also denote ∂_{x_j} by ∂_j for convenience.

Consider the orthogonal decompostion $L^2(\mathbb{R}^3)=H_\sigma\oplus G_\sigma$ with

$$H_\sigma=\{u\in L^2(\mathbb{R}^3):\nabla\cdot u=0 \text{ in } \mathbb{R}^3\},\qquad G_\sigma=\{\nabla\psi:\psi\in H^1(\mathbb{R}^3)\}. \tag{1.8}$$

Let $\mathcal{P}$ be the projection onto H_σ and $\mathcal{Q}=I-\mathcal{P}$. It is widely known that $\mathcal{P}\in\mathscr{L}(H^m,H^m)$ for any $m\ge 0$.

1.4 Our results

We have the following uniform stability and convergence result.

Theorem 1.1 (Uniform existence). *Assume that the initial data* $U_0^\varepsilon=(r_0^\varepsilon,u_0^\varepsilon,H_0^\varepsilon,\Theta_0^\varepsilon)\in H^3(\mathbb{R}^3)$ *satisfies that for all* $\varepsilon\in(0,1]$,

$$\|U_0^\varepsilon\|_3\le M_0 \tag{1.9}$$

for some constant $M_0>0$. *Then there exist constants* $T>0$ *and* $M>0$ *independent of* ε *such that for each* $\varepsilon\in(0,1]$ *the initial value problem* (1.4), (1.6) *has a classical solution* U^ε *on* $[0,T]$ *satisfying the following uniform estimate*

$$\sup_{t\in[0,T]}\|U^\varepsilon(t)\|_3\le M. \tag{1.10}$$

Next, we state the convergence result. We will prove that the fast parts r^ε and $\mathcal{Q}u^\varepsilon$ converge to 0 by using the dispersive properties of the fast equations.

Theorem 1.2 (Incompressible limit in the whole space). *Let the assumption of Theorem* 1.1 *holds. We assume that the intitial data* $(r_0^\varepsilon,u_0^\varepsilon,H_0^\varepsilon,\Theta_0^\varepsilon)$ *converge to* $(\bar{r},\overline{u},\overline{H},\overline{\Theta})$ *in* $H^3(\mathbb{R}^3)$ *as* $\varepsilon\to 0$. *Then there exists a function* $U^0=(r^0,u^0,H^0,\Theta^0)$, *such that the solution of problem* (1.4) *and* (1.6) *satisfies*

$$(r^\varepsilon,u^\varepsilon,H^\varepsilon,\Theta^\varepsilon)\to(r^0,u^0,H^0,\Theta^0)$$

weakly- in* $L^\infty([0,T];H^3(\mathbb{R}^3))$ *and strongly in* $C_{loc}((0,T]\times\mathbb{R}^3)$ *as* $\varepsilon\to 0$. *Moreover,* (u^0,H^0) *is the solution in* $C([0,T];H^3(\mathbb{R}^3))$ *of the incompressible MHD Eqs.* (1.7) *with initial data* $(\mathcal{P}\overline{u},\overline{H})$ *and* $\overline{\rho}=\rho(1,1)$.

In the present paper, the incompressible limit of ideal nonisentropic MHD equations with general initial data in $\mathbb{R}^3$ will be considered. It needs to be admitted that the problem becomes even more complex when considering the case of large entropy variation. To be precise, it is difficult to show this cancelation due to the strong coupling of hydrodynamic motion and magnetic field for the large entropy variation case since the coefficients a and ρ depend on $(\epsilon r^\epsilon, S^\epsilon)$. In order to overcome the above difficulty, Jiang et al. [12] added some additional restrictions to obtain the uniform estimates and used the original ideas of Métivier and Schochet [16] to prove the convergence of solutions. Moreover, Li and Zhang [13] obtained the stronger convergence result than [12] where they considered the well-prepared initial data. Compared with [12,13], we no longer need additional restrictions of function space and well-prepared initial data, when we consider small variation of entropy in the sense of (1.3). Moreover, we get a stronger convergence result than [15]. Thanks to the fact that A_0 only depends on $\varepsilon U^\varepsilon$, we can perform energy estimates to (1.4) as in [1, 17]. However, the general initial data will lead to $\|\partial_t U^\varepsilon\|_2$ being $O(\varepsilon^{-1})$, which implies that we haven't got any uniform bounds for some norm of the first order time derivative of solution which is important for the convergence theory. To prove the convergence of solution, we need to use the original ideas of Ukai [18].

This paper is arranged as follows. In Section 2, we shall perform energy estimates to (1.4), and then give the proof Theorem 1.1. Finally, we shall prove Theorems 1.2 in Section 3.

2 Uniform estimates

For fixed $\varepsilon > 0$, there exists a local in time unique classical solution of (1.4) and (1.6) (see [19, Theorem 2.1]). The key point in the proof of Theorem 1.1 is to establish the uniform estimate (1.10). Throughout this section U^ε will be denoted by U, and the corresponding superscript ε used in other notation is omitted for simplicity of presentation.

We can perform energy estimates to (1.4) as follows. For any multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, applying D^α with $|\alpha| \le 3$ to (1.4), we deduce that

$$\begin{aligned}
&a(\partial_t D^\alpha r + u \cdot \nabla D^\alpha r) + \frac{1}{\varepsilon} \nabla \cdot D^\alpha u = \mathscr{C}_r, \\
&\rho(\partial_t D^\alpha u + u \cdot \nabla D^\alpha u) + \frac{1}{\varepsilon} \nabla D^\alpha r + \frac{1}{2} \nabla D^\alpha |H|^2 - H \cdot \nabla D^\alpha H = \mathscr{C}_u, \\
&\partial_t D^\alpha H + u \cdot \nabla D^\alpha H - H \cdot \nabla D^\alpha u + H \nabla \cdot D^\alpha u = \mathscr{C}_H, \\
&\partial_t D^\alpha \Theta + u \cdot \nabla D^\alpha \Theta = \mathscr{C}_\Theta,
\end{aligned} \tag{2.1}$$

where the commutators are given by

$$\begin{aligned}
&\mathscr{C}_r = -[D^\alpha, a]\partial_t r - [D^\alpha, a u \cdot \nabla] r, \\
&\mathscr{C}_u = -[D^\alpha, \rho]\partial_t u - [D^\alpha, \rho u \cdot \nabla] u - [D^\alpha, H \times](\nabla \times H),
\end{aligned}$$

$$\mathscr{C}_H = -[D^\alpha, u\cdot\nabla]H + [D^\alpha, H\cdot\nabla]u - [D^\alpha, H]\nabla\cdot u,$$
$$\mathscr{C}_\Theta = -[D^\alpha, u\cdot\nabla]\Theta.$$

Multiplying (2.1) by $D^\alpha U$, integrating over $\mathbb{R}^3$, and integrating by parts give that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\left(a|D^\alpha r|^2+\rho|D^\alpha u|^2+|D^\alpha H|^2+|D^\alpha\Theta|^2\right)\mathrm{d}x\\
=&\frac{1}{2}\int_{\mathbb{R}^3}\left((\partial_t a+\nabla\cdot(au))|D^\alpha r|^2+\nabla\cdot u|D^\alpha H|^2+\nabla\cdot u|D^\alpha\Theta|^2\right)\mathrm{d}x\\
&\quad+\int_{\mathbb{R}^3}\left(\frac{1}{2}\nabla D^\alpha|H|^2\cdot D^\alpha u+(H\cdot D^\alpha H)\nabla\cdot D^\alpha u\right)\mathrm{d}x\\
&\quad+\int_{\mathbb{R}^3}((H\cdot\nabla D^\alpha u)\cdot D^\alpha H+(H\cdot\nabla D^\alpha H)\cdot D^\alpha u)\mathrm{d}x\\
&\quad+\int_{\mathbb{R}^3}(D^\alpha r\mathscr{C}_r+D^\alpha u\cdot\mathscr{C}_u+D^\alpha H\cdot\mathscr{C}_H+D^\alpha\Theta\cdot\mathscr{C}_\Theta)\mathrm{d}x\\
=:&J_1+J_2+J_3+J_4,
\end{aligned}\tag{2.2}$$

where the singular terms vanish by Stokes formula. Next, we will control each term on the RHS of (2.2). Since a is smooth functions of $(\varepsilon r, \varepsilon\Theta)$, by the Sobolev inequality, we have

$$\|\partial_t a\|_{L^\infty}+\|\nabla\cdot(au)\|_{L^\infty}\le F(\|U\|_3).$$

Thus, it is easy to find that J_1 can be bounded from above by

$$|J_1|| \le \frac{1}{2}(\|\partial_t a\|_{L^\infty}+\|\nabla\cdot(au)\|_{L^\infty}\|\nabla\cdot u\|_{L^\infty})\|D^\alpha U\|^2\le F(\|U\|_3).$$

For J_2 and J_3, integrating by parts, one obtains

$$\begin{aligned}
J_2=&\int_{\mathbb{R}^3}\left(-H\cdot D^\alpha H-\sum_{\beta\le\alpha,|\beta|\ge1}C_{\alpha,\beta}D^\beta H\cdot D^{\alpha-\beta}H+H\cdot D^\alpha H\right)\nabla\cdot D^\alpha u\mathrm{d}x\le F(\|U\|_3),\\
J_3=&\int_{\mathbb{R}^3}(-(H\cdot\nabla D^\alpha u)\cdot D^\alpha H+(H\cdot\nabla D^\alpha u)\cdot D^\alpha H)\mathrm{d}x=0.
\end{aligned}$$

Finally, we study the commutator estimate J_4. We just give the estimate of $\mathscr{C}_r$ here, and the other terms can be controlled in the same manner. Recalling the expression of the commutator $\mathscr{C}_r$, one sees

$$\begin{aligned}
\|\mathscr{C}_r\|\le&\|[D^\alpha,a]\partial_t r\|+\|[D^\alpha,au\cdot\nabla]r\|\\
\lesssim&\sum_{\beta\le\alpha,|\beta|\ge1}\|D^\beta a\cdot D^{\alpha-\beta}\partial_t r\|+\sum_{\beta\le\alpha,|\beta|\ge1}\|D^\beta(au)\cdot D^{\alpha-\beta}\nabla r\|\\
:=&R_1+R_2.
\end{aligned}\tag{2.3}$$

We can use standard commutator estimates to bound R_2 as follows:

$$R_2 \le F(\|U\|_3). \tag{2.4}$$

We use the first equation in (1.4) to replace $\partial_t r$ in R_1 by $\partial_t r = -\frac{1}{\varepsilon a}\nabla\cdot u - u\cdot\nabla r$, where, clearly, the singular term appears. Fortunately, considering that a is a smooth solution of $(\varepsilon r, \varepsilon\Theta)$ and $|\beta|\ge 1$, we can deduce that

$$\begin{aligned} R_1 \lesssim & \varepsilon^{-1} \sum_{\beta\le\alpha,|\beta|\ge 1} \left\| D^\beta a \cdot D^{\alpha-\beta}\left(\frac{1}{a}\nabla\cdot u\right)\right\| \\ & + \sum_{\beta\le\alpha, j+|\beta|\ge 1} \left\| D^\beta a\cdot D^{\alpha-\beta}(u\cdot\nabla r)\right\| \le F(\|U\|_3). \end{aligned} \tag{2.5}$$

Summing up (2.3)-(2.5), we conclude

$$\int_{\mathbb{R}^3} D^\alpha r \mathscr{C}_r \mathrm{d}x \le \|D^\alpha r\| \|\mathscr{C}_r\| \le F(\|U\|_3). \tag{2.6}$$

Since the other terms in J_4 can be estimated in a similar fashion, actually we find that J_4 enjoys an estimate similar to (2.6). Substituting the estimates for J_1-J_4 into (2.2) and taking summation with respect to α, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|U\|_3^2 \le F(\|U\|_3).$$

Thus, we can choose sufficiently small $T>0$ satisfying

$$\sup_{t\in[0,T]} \|U(t)\|_3^2 \le C(M_0). \tag{2.7}$$

Then, the uniform estimate (1.10) follows from (2.7).

3 Convergence results in the whole space

In this section, we introduce new variables by $\dot{r}^\varepsilon = (\frac{\bar{\rho}}{\bar{a}})^{\frac{1}{2}} r^\varepsilon$, and we still denote $\dot{r}^\varepsilon$ by r^ε for convenience. Then we can rewrite Eq. (1.4) into the following form

$$\begin{aligned} & \partial_t r^\varepsilon + u^\varepsilon\cdot\nabla r^\varepsilon + \frac{(\frac{\bar{a}}{\bar{\rho}})^{\frac{1}{2}}}{\varepsilon a(\varepsilon r^\varepsilon, \varepsilon\Theta^\varepsilon)}\nabla\cdot u^\varepsilon = 0, \\ & \partial_t u^\varepsilon + u^\varepsilon\cdot\nabla u^\varepsilon + \frac{(\frac{\bar{\rho}}{\bar{a}})^{\frac{1}{2}}}{\varepsilon\rho(\varepsilon r^\varepsilon,\varepsilon\Theta^\varepsilon)}\nabla r^\varepsilon = -\frac{1}{\rho(\varepsilon r^\varepsilon,\varepsilon\Theta^\varepsilon)} H^\varepsilon\times(\nabla\times H^\varepsilon), \\ & \partial_t H^\varepsilon + u^\varepsilon\cdot\nabla H^\varepsilon = H\cdot\nabla u^\varepsilon - H\nabla\cdot u^\varepsilon, \qquad \nabla\cdot H^\varepsilon = 0, \\ & \partial_t\Theta^\varepsilon + u^\varepsilon\cdot\nabla\Theta^\varepsilon = 0. \end{aligned} \tag{3.1}$$

We set $\rho(1,1)=\bar{\rho}$, $a(1,1)=\bar{a}$. Then we have the compact form:

$$\partial_t U^\varepsilon - \mathcal{B}U^\varepsilon = J(U^\varepsilon,\nabla U^\varepsilon), \tag{3.2}$$

where

$$\mathcal{B} := -\frac{1}{\varepsilon}\left(\frac{1}{\bar{a}\bar{\rho}}\right)^{\frac{1}{2}} \begin{pmatrix} 0 & \nabla\cdot & 0_{1\times4} \\ \nabla & 0_{3\times3} & 0_{3\times4} \\ 0_{4\times1} & 0_{4\times3} & 0_{4\times4} \end{pmatrix},$$

$$J(U^\varepsilon,\nabla U^\varepsilon) = \begin{pmatrix} -u^\varepsilon\cdot\nabla r^\varepsilon + (\frac{\bar{a}}{\bar{\rho}})^{\frac{1}{2}} \frac{\bar{a}^{-1}-a^{-1}}{\varepsilon}\nabla\cdot u^\varepsilon \\ \rho^{-1}(\nabla\times H^\varepsilon)\times H^\varepsilon - u^\varepsilon\cdot\nabla u^\varepsilon + (\frac{\bar{\rho}}{\bar{a}})^{\frac{1}{2}}\frac{\bar{\rho}^{-1}-\rho^{-1}}{\varepsilon}\nabla r^\varepsilon \\ H^\varepsilon\cdot\nabla u^\varepsilon - H^\varepsilon\nabla\cdot u^\varepsilon - u^\varepsilon\cdot\nabla H^\varepsilon \\ -u^\varepsilon\cdot\nabla\Theta^\varepsilon \end{pmatrix}.$$

We will study the group $e^{t\mathcal{B}}$ generated by operator $\mathcal{B}$. Using the Fourier transform

$$(\mathcal{F}u)(\xi) = \hat{u}(\xi) = (2\pi)^{-\frac{3}{2}}\int_{\mathbb{R}^3} e^{-ix\cdot\xi}u(x)\mathrm{d}x,$$

we readily find the solution operator $e^{t\mathcal{B}}$ in the form $e^{t\mathcal{B}} = \mathcal{F}^{-1}e^{t\hat{\mathcal{B}}}\mathcal{F}$, where $\hat{\mathcal{B}}$ is defined by

$$\hat{\mathcal{B}} := -\frac{\mathrm{i}}{\varepsilon}\left(\frac{1}{\bar{a}\bar{\rho}}\right)^{\frac{1}{2}} \begin{pmatrix} 0 & \xi\cdot & 0_{1\times4} \\ \xi & 0_{3\times3} & 0_{3\times4} \\ 0_{4\times1} & 0_{4\times3} & 0_{4\times4} \end{pmatrix}.$$

3.1 Spectral analysis of the solution operator

The eigenvalue problem for the operator $\mathcal{B}$ reduces to the algebraic eigenvalue problem for $\hat{\mathcal{B}}$ that we summarize below:

- *Eigenvalue problem:*

$$\hat{\mathcal{B}}w_\xi^{(\alpha)} = \lambda_\xi^{(\alpha)} w_\xi^{(\alpha)};$$

- *Eigenvlues:*

$$\lambda_\xi^{(1)} = -\frac{\mathrm{i}(\bar{a}\bar{\rho})^{-\frac{1}{2}}|\xi|}{\varepsilon}, \quad \lambda_\xi^{(2)} = \frac{\mathrm{i}(\bar{a}\bar{\rho})^{-\frac{1}{2}}|\xi|}{\varepsilon}, \quad \lambda_\xi^{(\alpha)} = 0, \qquad 3\le\alpha\le8;$$

- *Right eigenvectors:*

$$w_\xi^{(1)} = (\bar{a}\bar{\rho})^{\frac{1}{2}}\begin{pmatrix} 1 \\ \frac{\xi}{|\xi|} \\ 0 \end{pmatrix}, \quad w_\xi^{(2)} = (\bar{a}\bar{\rho})^{\frac{1}{2}}\begin{pmatrix} -1 \\ \frac{\xi}{|\xi|} \\ 0 \end{pmatrix}, \quad w_\xi^{(3)} = \frac{1}{|\xi|}\begin{pmatrix} 0 \\ \xi^\perp \\ 0 \end{pmatrix},$$

$$w_\xi^{(\alpha)} = e_\alpha, \quad 4\le\alpha\le8.$$

Then we have

$$\left(w_\xi^{(\alpha)}\right)^* w_\xi^{(\alpha)} = 1, \qquad \sum_\alpha w_\xi^{(\alpha)} \left(w_\xi^{(\alpha)}\right)^* = I_8,$$

where

$$\left(w_\xi^{(1)}\right)^* = \frac{1}{2}(\bar{a}\bar{\rho})^{-\frac{1}{2}} \begin{pmatrix} 1 & \frac{\xi^{\rm T}}{|\xi|} & 0 \end{pmatrix}, \quad \left(w_\xi^{(2)}\right)^* = \frac{1}{2}(\bar{a}\bar{\rho})^{-\frac{1}{2}} \begin{pmatrix} -1 & \frac{\xi^{\rm T}}{|\xi|} & 0 \end{pmatrix},$$
$$\left(w_\xi^{(\alpha)}\right)^* = \left(w_\xi^{(\alpha)}\right)^{\rm T}, \qquad 3 \le \alpha \le 8.$$

Moreover, it is easy to verify that $\mathcal{P}$ defined by $\mathcal{P}u = \mathcal{F}^{-1}\left(|\xi|^{-2}\xi^\perp\left(\xi^\perp\right)^{\rm T}\hat{u}(\xi)\right)$ is the projection onto H_σ. Using spectral analysis, we obtain

$$\hat{\mathcal{B}} = \sum_\alpha \lambda_\xi^{(\alpha)} w_\xi^{(\alpha)} \left(w_\xi^{(\alpha)}\right)^*, \quad e^{t\hat{\mathcal{B}}} = \sum_\alpha e^{\lambda_\xi^{(\alpha)} t} w_\xi^{(\alpha)} \left(w_\xi^{(\alpha)}\right)^*,$$

$$e^{t\mathcal{B}} = \sum_{3\le\alpha\le 8} \mathcal{F}^{-1}\left(w_\xi^{(\alpha)}\left(w_\xi^{(\alpha)}\right)^* \mathcal{F}\cdot\right) + \sum_{\alpha=1,2} \mathcal{F}^{-1}\left(e^{\lambda_\xi^{(\alpha)} t} w_\xi^{(\alpha)}\left(w_\xi^{(\alpha)}\right)^* \mathcal{F}\cdot\right) \tag{3.3}$$
$$=: \mathcal{U}_1 + \mathcal{U}_2^\varepsilon(t), \tag{3.4}$$

where

$$\mathcal{U}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \mathcal{P} & 0 \\ 0 & 0 & I_4 \end{pmatrix}, \qquad \mathcal{U}_2^\varepsilon(t)U = \sum_{\alpha=1,2} \mathcal{F}^{-1}\left(e^{\lambda_\xi^{(\alpha)} t} w_\xi^{(\alpha)}\left(w_\xi^{(\alpha)}\right)^* \hat{U}(\xi)\right),$$

and $\mathcal{U}_1\mathcal{U}_2^\varepsilon = \mathcal{U}_2^\varepsilon\mathcal{U}_1 = 0$.

The L^∞ decay property of $\mathcal{U}_2^\varepsilon(t)$ has been studied by [18], in which the author used Strichartz's estimates.

Lemma 3.1 ([18]). *For any $\ell > \frac{3}{2}$, there is a constant $C \ge 0$ such that for all $\varepsilon \in (0,1)$, $t \ge 0$ and $U \in H^\ell(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$ such that*

$$\|\mathcal{U}_2^\varepsilon(t)U\|_{L^\infty} \le C|\varepsilon^{-1}t|^{-\delta}\|U\|_{L^1}^\delta \|U\|_\ell^{1-\delta}, \tag{3.5}$$

with $\delta = 1 - \frac{4}{2\ell+1}$. Moreover if $U \in H^\ell(\mathbb{R}^3)$, then for any $t^* > 0$,

$$\sup_{t\ge t^*} \|\mathcal{U}_2^\varepsilon(t)U\|_{L^\infty} \to 0 \text{ as } \varepsilon \to 0. \tag{3.6}$$

3.2 Convergence theorem

The uniform estimate (1.10) implies, after extracting a subsequence, the following convergence:

$$U^\varepsilon \to U^0 \quad \text{weakly-* in } L^\infty([0,T]; H^3(\mathbb{R}^3)), \tag{3.7}$$

where $U^0=(r^0,u^0,H^0,\Theta^0)\in L^\infty([0,T];H^3(\mathbb{R}^3))$. Recalling (3.2), we have

$$U^\varepsilon(t)=e^{t\mathcal{B}}U_0^\varepsilon+\int_0^t e^{(t-s)\mathcal{B}}J^\varepsilon(s)\mathrm{d}s, \tag{3.8}$$

where $J^\varepsilon(s)=J(U^\varepsilon(s),\nabla U^\varepsilon(s))$.

Lemma 3.2. *There is a constant $K>0$ independent of $\varepsilon\in(0,1)$ and $t\in[0,T]$, and it holds that*

$$\|J^\varepsilon(t)\|_{L^1}+\|J^\varepsilon(t)\|_2\le K. \tag{3.9}$$

Proof. Recall that

$$J(U^\varepsilon,\nabla U^\varepsilon)=\begin{pmatrix} -u^\varepsilon\cdot\nabla r^\varepsilon+(\frac{\bar a}{\bar\rho})^{\frac12}\frac{\bar a^{-1}-a^{-1}}{\varepsilon}\nabla\cdot u^\varepsilon \\ \rho^{-1}(\nabla\times H^\varepsilon)\times H^\varepsilon-u^\varepsilon\cdot\nabla u^\varepsilon+(\frac{\bar\rho}{\bar a})^{\frac12}\frac{\bar\rho^{-1}-\rho^{-1}}{\varepsilon}\nabla r^\varepsilon \\ H^\varepsilon\cdot\nabla u^\varepsilon-H^\varepsilon\nabla\cdot u^\varepsilon-u^\varepsilon\cdot\nabla H^\varepsilon \\ -u^\varepsilon\cdot\nabla\Theta^\varepsilon \end{pmatrix},$$

and the facts

$$\bar a^{-1}-a^{-1}=O(\varepsilon),\qquad \bar\rho^{-1}-\rho^{-1}=O(\varepsilon).$$

By Schwarz inequality

$$\|J^\varepsilon\|_{L^1}\le K\left(\|U^\varepsilon\|+\left\|\frac{\bar a^{-1}-a^{-1}}{\varepsilon}\right\|+\left\|\frac{\bar\rho^{-1}-\rho^{-1}}{\varepsilon}\right\|\right)\|\nabla U^\varepsilon\|\le K. \tag{3.10}$$

Further, by Moser-type calculus inequalities ([19]), we have

$$\|J^\varepsilon\|_2\le K(1+\|U^\varepsilon\|_2)^2\|U^\varepsilon\|_3\le K.$$

This completes the proof of this lemma. □

We decompose U^ε according to the decomposition (3.4):

$$U^\varepsilon=\mathcal{U}_1U^\varepsilon+\mathcal{U}_2^\varepsilon U^\varepsilon, \tag{3.11}$$

$$\mathcal{U}_1U^\varepsilon(t)=\begin{pmatrix} 0 \\ \mathcal{P}u_0^\varepsilon+\int_0^t\mathcal{P}G^\varepsilon(s)\mathrm{d}s \\ H^\varepsilon \\ \Theta^\varepsilon \end{pmatrix}, \tag{3.12}$$

$$\mathcal{U}_2^\varepsilon(t)U^\varepsilon(t)=\mathcal{U}_2^\varepsilon(t)U_0^\varepsilon+\int_0^t\mathcal{U}_2^\varepsilon(t-s)J^\varepsilon(s)\mathrm{d}s, \tag{3.13}$$

where

$$G^\varepsilon(s)=-U^\varepsilon(s)\cdot\nabla u^\varepsilon(s)+(\frac{\bar\rho}{\bar a})^{\frac12}\frac{\bar\rho^{-1}-\rho^{-1}}{\varepsilon}\nabla r^\varepsilon(s)-\frac{1}{\rho(s)}H^\varepsilon(s)\times(\nabla\times H^\varepsilon(s)). \tag{3.14}$$

We will prove the following estimate, in order to get the convergence of $\mathcal{U}_1U^\varepsilon$.

Lemma 3.3. *There is a constant $K>0$ independent of $\varepsilon\in(0,1)$ and $t\in[0,T]$, such that it holds that*

$$\|\mathcal{U}_1U^\varepsilon\|_3+\|\mathcal{U}_1\partial_tU^\varepsilon\|_2\le K. \tag{3.15}$$

Proof. Since the decomposition (3.11) is orthogonal, we obtain

$$\|U^\varepsilon\|_3^2=\|\mathcal{U}_1U^\varepsilon\|_3^2+\|\mathcal{U}_2^\varepsilon U^\varepsilon\|_3^2.$$

Then by uniform estimate of U^ε,

$$\|\mathcal{U}_1U^\varepsilon\|_3\le\|U^\varepsilon\|_3\le K. \tag{3.16}$$

Applying ∂_t to (3.12), we have

$$\mathcal{U}_1\partial_tu^\varepsilon=\mathcal{P}G^\varepsilon,\quad \mathcal{U}_1\partial_t\begin{pmatrix}H^\varepsilon\\ \Theta^\varepsilon\end{pmatrix}=\begin{pmatrix}H^\varepsilon\cdot\nabla u^\varepsilon-H^\varepsilon\nabla\cdot u^\varepsilon-u^\varepsilon\cdot\nabla H^\varepsilon\\ -u^\varepsilon\cdot\nabla\Theta^\varepsilon\end{pmatrix}.$$

Then by Lemma 3.2, and uniform estimate of U^ε, we have

$$\|\mathcal{U}_1\partial_tu^\varepsilon\|_2\le\|\mathcal{P}G^\varepsilon\|_2\le\|G^\varepsilon\|_2\le\|J^\varepsilon\|_2\le K,\qquad \|\mathcal{U}_1\partial_tH^\varepsilon\|_2+\|\mathcal{U}_1\partial_t\Theta^\varepsilon\|_2\le K.$$

This completes the proof of the lemma. □

Furthermore, by Lemma 3.1, we will prove $\mathcal{U}_2^\varepsilon(t)U^\varepsilon(t)$ has the L^∞ decay.

Lemma 3.4. *For any $t^*>0$, it holds that*

$$\sup_{t\ge t^*}\|\mathcal{U}_2^\varepsilon U^\varepsilon(t)\|_{L^\infty}\to 0\quad\text{as }\ \varepsilon\to 0. \tag{3.17}$$

Proof. Lemma 3.1 can be applied to (3.13) with $\ell=2$ and $\delta=\frac{1}{5}$. By Lemma 3.2 we have

$$\begin{aligned}\|\mathcal{U}_2^\varepsilon U^\varepsilon(t)\|_{L^\infty}&\le\|\mathcal{U}_2^\varepsilon U_0^\varepsilon\|_{L^\infty}+K\varepsilon^\delta\int_0^t|t-s|^{-\delta}\|J^\varepsilon(s)\|_{L^1}^\delta\|J^\varepsilon(s)\|_2^{1-\delta}\mathrm{d}s\\ &\le\|\mathcal{U}_2^\varepsilon U_0^\varepsilon\|_{L^\infty}+K\varepsilon^\delta.\end{aligned} \tag{3.18}$$

Let $\varepsilon\to 0$ and use Lemma 3.1 again, we finally complete the proof. □

By Lemma 3.3 and Arzelà-Ascoli theorem, up to a subsequence

$$\mathcal{U}_1U^\varepsilon\to U^*=(0,u^*,H^*,\Theta^*)\quad\text{strongly in } C_{\rm loc}([0,T];H^{s'}(\mathbb{R}^3)),\qquad s'<3. \tag{3.19}$$

Taking account of Lemma 3.4, we then conclude that

$$U^\varepsilon\to U^*\quad\text{strongly in } C_{\rm loc}((0,T]\times\mathbb{R}^3). \tag{3.20}$$

By weak convergence (3.7) we have

$$(r^0,u^0,H^0,\Theta^0)=(0,u^*,H^*,\Theta^*) \tag{3.21}$$

and $\mathcal{P}u^0=u^0$. This proves

$$\nabla\cdot u^0=0. \tag{3.22}$$

Passing to the limit in the equations for H^ε, we see that the limits H^0 satisfy

$$\partial_t H^0+u^0\cdot\nabla H^0=H^0\cdot\nabla u^0+H^0\nabla\cdot u^0, \qquad \nabla\cdot H^0=0. \tag{3.23}$$

Passing to the limit in G^ε we see that

$$G^\varepsilon\to -u^0\cdot\nabla u^0+\frac{1}{\bar\rho}H^0\cdot\nabla H^0 \quad \text{weakly-* in } L^\infty([0,T];H^2(\mathbb{R}^3)). \tag{3.24}$$

Then we can take the limit on both sides of the second equation in (3.12), i.e.,

$$\mathcal{P}u^\varepsilon(t)=\mathcal{P}u_0^\varepsilon+\int_0^t \mathcal{P}G^\varepsilon(s)\mathrm{d}s,$$

and get

$$u^0(t)=\mathcal{P}\overline{u}+\int_0^t\left(-u^0(s)\cdot\nabla u^0(s)+\frac{1}{\bar\rho}H^0(s)\cdot\nabla H^0(s)\right)\mathrm{d}s.$$

Hence

$$\partial_t u^0=\mathcal{P}\left(-u^0\cdot\nabla u^0+\frac{1}{\bar\rho}H^0\cdot\nabla H^0\right), \quad u_0^0=\mathcal{P}\overline{u}. \tag{3.25}$$

Now, by (3.22), (3.23) and (3.25), (u^0,H^0) is the solution in $C([0,T];H^3(\mathbb{R}^3))$ of following incompressible MHD equations for a suitable pressure fuction $\nabla\pi\in L^\infty([0,T];H^2(\mathbb{R}^3))$.

$$\begin{aligned}
&\bar\rho(\partial_t u^0+u^0\cdot\nabla u^0)+\nabla\pi-H^0\cdot\nabla H^0=0,\\
&\partial_t H^0+u^0\cdot\nabla H^0-H^0\cdot\nabla u^0=0,\\
&\nabla\cdot u^0=0, \qquad \nabla\cdot H^0=0,\\
&(u^0,H^0)(t=0)=(\mathcal{P}\overline{u},\overline{H}).
\end{aligned} \tag{3.26}$$

Moreover, the uniqueness of the limit function implies that the convergence holds as $\varepsilon\to 0$ without restricting to a subsequence. This completes the proof of Theorem 1.2.

Acknowledgments

The authors would like to thank Professor Qiangchang Ju for helpful discussions during the preparation of this paper. The authors are supported by the NSFC (Grants 12131007 and 12071044).

References

[1] Klainerman S., Majda A., Singular limits of quasilinear hyperbolic systems with large parameters and the incompressible limit of compressible fluids. *Commun. Pure Appl. Math.*, **34** (1981), 481–524.

[2] Li Y., Convergence of the compressible magnetohydrodynamic equations to incompressible magnetohydrodynamic equations. *J. Differential Equ.*, **252** (2012), 2725–2738.

[3] Hu X., Wang D., Low Mach number limit of viscous compressible magnetohydrodynamic flows. *SIAM J. Math. Anal.*, **41** (2009), 1272–1294.

[4] Jiang S., Ju Q. and Li F., Incompressible limit of the compressible magnetohydrodynamic equations with vanishing viscosity coefficients. *SIAM J. Math. Anal.*, **42** (2010), 2539–2553.

[5] Jiang S., Ju Q. and Li F., Incompressible limit of the compressible magnetohydrodynamic equations with periodic boundary conditions. *Commun. Math. Phys.*, **297** (2010), 371–400.

[6] Feireisl E., Novotný A. and Sun Y., Dissipative solutions and the incompressible inviscid limits of the compressible magnetohydrodynamic system in unbounded domains. *Discrete Contin. Dyn. Syst.*, **34** (2014), 121–143.

[7] Guo L., Li F. and Xie F., Asymptotic limits of the isentropic compressible viscous magnetohydrodynamic equations with Navier-slip boundary conditions. *J. Differential. Equ.*, **267** (2019), 6910–6957.

[8] Jiang S., Ju Q. and Li F., Low Mach number limit for the multi-dimensional full magnetohydrodynamic equations. *Nonlinearity.*, **25** (2012), 1351.

[9] Cui W., Ou Y. and Ren D., Incompressible limit of full compressible magnetohydrodynamic equations with well-prepared data in 3-D bounded domains. *J. Math. Anal. Appl.*, **427** (2015), 263–288.

[10] Ou Y., Yang L., Incompressible limit of non-isentropic compressible magnetohydrodynamic equations with zero magnetic diffusivity in bounded domains. *Nonlinear Anal. Real World Appl.*, **49** (2019), 1–23.

[11] Jiang S., Ju Q., Li F. and Xin Z., Low Mach number limit for the full compressible magnetohydrodynamic equations with general initial data. *Adv. Math.*, **259** (2014), 384–420.

[12] Jiang S., Ju Q. and Li F., Incompressible limit of the nonisentropic ideal magnetohydrodynamic equations. *SIAM J. Math. Anal.*, **48** (2016), 302–319.

[13] Li F., Zhang S., Low Mach number limit of the non-isentropic ideal magnetohydrodynamic equations. *J. Math. Fluid Mech.*, **23** (2021), 1–15.

[14] Schochet S., Fast singular limits of hyperbolic PDEs. *J. Differential. Equ.*, **114** (1994), 476–512.

[15] Meng F., Wang J., Incompressible limit of nonisentropic ideal magnetohydrodynamic equations with periodic boundary conditions. *Appl. Anal.*, **103** (2024), 1300–1313.

[16] Métivier G., Schochet S., The incompressible limit of the non-isentropic Euler equations. *Arch. Ration. Mech. Anal.*, **158** (2001), 61–90.

[17] Klainerman S., Majda A., Compressible and incompressible fluids. *Commun. Pure Appl. Math.*, **35** (1982), 629–651.

[18] Ukai S., The incompressible limit and the initial layer of the compressible euler equation. *J. Math. Kyoto Univ.*, **26** (1986), 323–331.

[19] Majda A., Compressible Fluid Flow and Systems of Conservation Laws in Several Space Variables. Springer-Verlag, New York, 1984.