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Existence of Solutions to a Generalized Self-Dual Chern-Simons System on Finite Graphs

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Abstract. We study a system of equations arising in the Chern-Simons model on finite graphs. Using the iteration scheme and the upper and lower solutions method, we get existence of solutions in the non-critical case. The critical case is dealt with by priori estimates. Our results generalize those of Huang et al. (Journal of Functional Analysis 281(10) (2021) Paper No. 109218).

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1 Introduction

The Chern-Simons models describe gauge fields governed by Chern-Simons type dynamics, and explain certain phenomena in the fields of particle physics, condensed matter physics and so on [1–3]. Some Chern-Simons models can be reduced to elliptic equations with exponential nonlinearities. Many studies were devoted to self-dual Chern-Simons equations including nonrelativistic and relativistic cases, Abelian and non-Abelian cases.

In this paper, we consider the following Chern-Simons system

$$\begin{cases} \Delta u = -\lambda \mathrm{e}^v H(\mathrm{e}^v) g(\mathrm{e}^u) + 4\pi \sum_{j=1}^{N_1} \delta_{p_j'}, \\ \Delta v = -\lambda \mathrm{e}^u G(\mathrm{e}^u) h(\mathrm{e}^v) + 4\pi \sum_{j=1}^{N_2} \delta_{p_j''}, \end{cases} \tag{1.1}$$

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on a finite graph, where $G>0$, $H>0$ are increasing, C^∞ functions in $[0,\infty)$; g and h are defined by $g(s^2)=\int_s^1 2sG(s^2)\mathrm{d}s$ and $h(s^2)=\int_s^1 2sH(s^2)\mathrm{d}s$, respectively; $\lambda>0$ is a constant; N_1 and N_2 are positive integers; δ_p is the Dirac delta mass at vertex p. The system (1.1) was proposed in [4] to study the $U(1)\times U(1)$ Chern-Simons model with a general Higgs potential. For the special case $G\equiv 1$ and $H\equiv 1$, the existence of solutions to the system (1.1) was obtained in [5,6], and the discrete form of (1.1) on finite graphs was investigated in [7]. For more results on discrete equations with exponential nonlinearities, one may refer to [8–16].

We write $G=(V,E)$ to denote a connected finite graph, where V and E represent vertices and edges, respectively. We assume the weight $\omega_{xy}>0$ on edge xy is symmetric. Let $\mu: V\to\mathbb{R}^+$ be a finite measure. For functions $u,v: V\to\mathbb{R}$, we define the μ-Laplace operator by

$$\Delta u(x)=\frac{1}{\mu(x)}\sum_{y\sim x}\omega_{xy}(u(y)-u(x)), \tag{1.2}$$

and let

$$\Gamma(u,v)=\frac{1}{2\mu(x)}\sum_{y\sim x}\omega_{xy}(u(y)-u(x))(v(y)-v(x)), \tag{1.3}$$

where $y\sim x$ means vertex y is adjacent to vertex x. Write

$$|\nabla u|(x)=\left(\frac{1}{2\mu(x)}\sum_{y\sim x}\omega_{xy}(u(y)-u(x))^2\right)^{\frac{1}{2}}.$$

For any function $f: V\to\mathbb{R}$, the integral of f over V is defined by

$$\int_V f\mathrm{d}\mu=\sum_{x\in V}\mu(x)f(x).$$

We define the Sobolev space as in the Euclidean case by

$$W^{1,2}(V)=\left\{u\,\middle|\, u: V\to\mathbb{R}, \int_V\left(|\nabla u|^2+u^2\right)\mathrm{d}\mu<+\infty\right\}.$$

We get the following results about the existence of maximal solutions.

Theorem 1.1. *There exists* $\lambda_c\geq\frac{4\pi\max\{N_1,N_2\}}{G(1)H(1)|V|}$ *such that*

(1) If $\lambda>\lambda_c$*, the system* (1.1) *admits a unique maximal solution* (u_λ,v_λ) *in the sense that if* (u'_λ,v'_λ) *is any other solution, then* $u_\lambda>u'_\lambda$, $v_\lambda>v'_\lambda$*. Moreover, if* $\lambda_1>\lambda_2>\lambda_c$*, then* $u_{\lambda_1}>u_{\lambda_2}$ *and* $v_{\lambda_1}>v_{\lambda_2}$.

(2) If $\lambda<\lambda_c$*, the system* (1.1) *admits no solution.*

(3) If $\lambda=\lambda_c$*, the system* (1.1) *admits a solution* (u_*,v_*) *which satisfies* $u_*<u_\lambda$ *and* $v_*<v_\lambda$ *if* $\lambda_c<\lambda$.