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Existence of Solutions to a Generalized Self-Dual Chern-Simons System on Finite Graphs

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Abstract. We study a system of equations arising in the Chern-Simons model on finite graphs. Using the iteration scheme and the upper and lower solutions method, we get existence of solutions in the non-critical case. The critical case is dealt with by priori estimates. Our results generalize those of Huang et al. (Journal of Functional Analysis 281(10) (2021) Paper No. 109218).

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1 Introduction

The Chern-Simons models describe gauge fields governed by Chern-Simons type dynamics, and explain certain phenomena in the fields of particle physics, condensed matter physics and so on [1–3]. Some Chern-Simons models can be reduced to elliptic equations with exponential nonlinearities. Many studies were devoted to self-dual Chern-Simons equations including nonrelativistic and relativistic cases, Abelian and non-Abelian cases.

In this paper, we consider the following Chern-Simons system

$$\begin{cases} \Delta u = -\lambda \mathrm{e}^{v} H(\mathrm{e}^{v}) g(\mathrm{e}^{u}) + 4\pi \sum_{j=1}^{N_1} \delta_{p_j'}, \\ \Delta v = -\lambda \mathrm{e}^{u} G(\mathrm{e}^{u}) h(\mathrm{e}^{v}) + 4\pi \sum_{j=1}^{N_2} \delta_{p_j''}, \end{cases} \tag{1.1}$$

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on a finite graph, where $G>0$, $H>0$ are increasing, C^∞ functions in $[0,\infty)$; g and h are defined by $g(s^2)=\int_s^1 2sG(s^2)\mathrm{d}s$ and $h(s^2)=\int_s^1 2sH(s^2)\mathrm{d}s$, respectively; $\lambda>0$ is a constant; N_1 and N_2 are positive integers; δ_p is the Dirac delta mass at vertex p. The system (1.1) was proposed in [4] to study the $U(1)\times U(1)$ Chern-Simons model with a general Higgs potential. For the special case $G\equiv 1$ and $H\equiv 1$, the existence of solutions to the system (1.1) was obtained in [5,6], and the discrete form of (1.1) on finite graphs was investigated in [7]. For more results on discrete equations with exponential nonlinearities, one may refer to [8–16].

We write $G=(V,E)$ to denote a connected finite graph, where V and E represent vertices and edges, respectively. We assume the weight $\omega_{xy}>0$ on edge xy is symmetric. Let $\mu:V\to\mathbb{R}^+$ be a finite measure. For functions $u,v:V\to\mathbb{R}$, we define the μ-Laplace operator by

$$\Delta u(x)=\frac{1}{\mu(x)}\sum_{y\sim x}\omega_{xy}(u(y)-u(x)), \tag{1.2}$$

and let

$$\Gamma(u,v)=\frac{1}{2\mu(x)}\sum_{y\sim x}\omega_{xy}(u(y)-u(x))(v(y)-v(x)), \tag{1.3}$$

where $y\sim x$ means vertex y is adjacent to vertex x. Write

$$|\nabla u|(x)=\left(\frac{1}{2\mu(x)}\sum_{y\sim x}\omega_{xy}(u(y)-u(x))^2\right)^{\frac{1}{2}}.$$

For any function $f:V\to\mathbb{R}$, the integral of f over V is defined by

$$\int_V f\,\mathrm{d}\mu=\sum_{x\in V}\mu(x)f(x).$$

We define the Sobolev space as in the Euclidean case by

$$W^{1,2}(V)=\left\{u\,\middle|\,u:V\to\mathbb{R},\int_V\left(|\nabla u|^2+u^2\right)\mathrm{d}\mu<+\infty\right\}.$$

We get the following results about the existence of maximal solutions.

Theorem 1.1. *There exists* $\lambda_c\geq\frac{4\pi\max\{N_1,N_2\}}{G(1)H(1)|V|}$ *such that*

(1) *If* $\lambda>\lambda_c$, *the system* (1.1) *admits a unique maximal solution* (u_λ,v_λ) *in the sense that if* (u'_λ,v'_λ) *is any other solution, then* $u_\lambda>u'_\lambda$, $v_\lambda>v'_\lambda$. *Moreover, if* $\lambda_1>\lambda_2>\lambda_c$, *then* $u_{\lambda_1}>u_{\lambda_2}$ *and* $v_{\lambda_1}>v_{\lambda_2}$.

(2) *If* $\lambda<\lambda_c$, *the system* (1.1) *admits no solution.*

(3) *If* $\lambda=\lambda_c$, *the system* (1.1) *admits a solution* (u_*,v_*) *which satisfies* $u_*<u_\lambda$ *and* $v_*<v_\lambda$ *if* $\lambda_c<\lambda$.

We also employ the iteration scheme as described in [4, 6, 17], while use different methods in the proof of the case (3) in Theorem 1.1. Our results generalize those of Huang et al. [7].

2 Proof of the main results

Let (u_0,v_0) be a solution to the system

$$\begin{cases} \Delta u = -\dfrac{4\pi N_1}{|V|} + 4\pi \displaystyle\sum_{j=1}^{N_1} \delta_{p'_j}, \\ \Delta v = -\dfrac{4\pi N_2}{|V|} + 4\pi \displaystyle\sum_{j=1}^{N_2} \delta_{p''_j}. \end{cases} \tag{2.1}$$

Set $u' = u_0 + u$ and $v' = v_0 + v$ if (u',v') is a solution to system (1.1). Substituting them into (1.1) gives

$$\begin{cases} \Delta u = -\lambda e^{v_0+v} H(e^{v_0+v}) g(e^{u_0+u}) + \dfrac{4\pi N_1}{|V|}, \\ \Delta v = -\lambda e^{u_0+u} G(e^{u_0+u}) h(e^{v_0+v}) + \dfrac{4\pi N_2}{|V|}. \end{cases} \tag{2.2}$$

We say that (u_-,v_-) is a lower solution of (2.2) if it satisfies

$$\begin{cases} \Delta u_- \geq -\lambda e^{v_0+v_-} H(e^{v_0+v_-}) g(e^{u_0+u_-}) + \dfrac{4\pi N_1}{|V|}, \\ \Delta v_- \geq -\lambda e^{u_0+u_-} G(e^{u_0+u_-}) h(e^{v_0+v_-}) + \dfrac{4\pi N_2}{|V|}. \end{cases} \tag{2.3}$$

Let $(u_1,v_1) = (-u_0,-v_0)$. We carry out the following iteration procedure

$$\begin{cases} (\Delta - K) u_{n+1} = -\lambda e^{v_0+v_n} H(e^{v_0+v_n}) g(e^{u_0+u_n}) - K u_n + \dfrac{4\pi N_1}{|V|}, \\ (\Delta - K) v_{n+1} = -\lambda e^{u_0+u_n} G(e^{u_0+u_n}) h(e^{v_0+v_n}) - K v_n + \dfrac{4\pi N_2}{|V|}. \end{cases} \tag{2.4}$$

Lemma 2.1. *Let $\{(u_n,v_n)\}$ be the sequence determined by (2.4). Then for any lower solution (u_-,v_-) of (2.2), there holds*

$$\begin{cases} u_1 > u_2 > \cdots > u_n > \cdots > u_-, \\ v_1 > v_2 > \cdots > v_n > \cdots > v_-. \end{cases} \tag{2.5}$$

Furthermore, if (2.4) has a lower solution, it admits a unique maximal solution (u_λ,v_λ) in the sense that if (u'_λ,v'_λ) is any other solution, then $u_\lambda > u'_\lambda$, $v_\lambda > v'_\lambda$.

Proof. We will prove it by the induction method. For $n=1$, by the iteration scheme, we have

$$\begin{cases}(\Delta-K)(u_2-u_1)=4\pi\sum_{j=1}^{N_1}\delta_{p'_j},\\ (\Delta-K)(v_2-v_1)=4\pi\sum_{j=1}^{N_2}\delta_{p''_j}.\end{cases} \tag{2.6}$$

Then the maximum principle, i.e., Lemma 4.1 in [17] indicates $u_2\le u_1$ and $v_2\le v_1$. Suppose that u_2-u_1 attains the maximum 0 at some $x_0\in V$. Then by (2.6), we obtain $\Delta(u_2-u_1)(x_0)\ge 0$. However, by (1.2), $\Delta(u_2-u_1)(x_0)\le 0$. Hence, $(u_2-u_1)(x)=(u_2-u_1)(x_0)=0$ if $x\sim x_0$, which yields $(u_2-u_1)(x)\equiv 0$ since G is connected. This leads to a contradiction with the inequality $(\Delta-K)(u_2-u_1)>0$ at p'_j. Therefore, $u_2<u_1$, and similarly, $v_2<v_1$. Now suppose that

$$\begin{cases}u_1>\cdots>u_n,\\ v_1>\cdots>v_n.\end{cases} \tag{2.7}$$

Choose $K>\lambda H(1)G(1)$. It is seen from (2.4) that

$$\begin{aligned}&(\Delta-K)(u_{n+1}-u_n)\\ =&-\lambda e^{v_0+v_n}H(e^{v_0+v_n})g(e^{u_0+u_n})+\lambda e^{v_0+v_{n-1}}H(e^{v_0+v_{n-1}})g(e^{u_0+u_{n-1}})-K(u_n-u_{n-1})\\ \ge&-\lambda H(1)\left(g(e^{u_0+u_n})-g(e^{u_0+u_{n-1}})\right)-K(u_n-u_{n-1})\\ =&\left(\lambda H(1)e^{\xi}G(e^{\xi})-K\right)(u_n-u_{n-1})\\ \ge&(\lambda H(1)G(1)-K)(u_n-u_{n-1})\\ >&0,\end{aligned}$$

where we have used the mean value theorem and $u_0+u_n\le\xi\le u_0+u_{n-1}$. Applying the same method as used in proving $u_2<u_1$, we obtain $u_{n+1}<u_n$. Hence, we get

$$u_1>\cdots>u_n>\cdots.$$

Similarly, there also holds

$$v_1>\cdots>v_n>\cdots.$$

Next we prove $u_n>u_-$ and $v_n>v_-$ for any n. For $n=1$, we derive that

$$\begin{aligned}\Delta(u_--u_1)\ge&-\lambda e^{v_0+v_-}H(e^{v_0+v_-})g(e^{u_0+u_-})+4\pi\sum_{j=1}^{N_1}\delta_{p'_j}\\ =&-\lambda e^{v_0+v_-}H(e^{v_0+v_-})\left[g(e^{u_0+u_-})-g(e^{u_0+u_1})\right]+4\pi\sum_{j=1}^{N_1}\delta_{p'_j}\end{aligned}$$

$$=\lambda e^{v_0+v_-}H(e^{v_0+v_-})e^{\xi}G(e^{\xi})(u_--u_1)+4\pi\sum_{j=1}^{N_1}\delta_{p'_j}, \tag{2.8}$$

where ξ lies between u_--u_1 and 0. Noting that G is finite, we have that there exists x_0 such that $(u_--u_1)(x_0)=\max_{x\in V}(u_--u_1)(x)$. Assuming that $(u_--u_1)(x_0)\geq 0$, then by (2.8) we have $\Delta(u_--u_1)(x_0)\geq 0$. Again, we have $\Delta(u_--u_1)(x_0)\leq 0$ by (1.2). Hence, $(u_--u_1)(x)=(u_--u_1)(x_0)$ if $x\sim x_0$, and $(u_--u_1)(x)\equiv(u_--u_1)(x_0)$ since G is connected, which contradicts (2.8) at p'_j. Hence, the assumption is not true and $u_-<u_1$. Similarly, $v_-<v_1$. For some $n\geq 1$, assume that $u_-<u_{n-1}$ and $v_-<v_{n-1}$. In view of (2.3) and (2.4), we arrive at

$$\begin{aligned}(\Delta-K)(u_--u_n)\geq&-\lambda e^{v_0+v_-}H(e^{v_0+v_-})g(e^{u_0+u_-})+\lambda e^{v_0+v_{n-1}}H(e^{v_0+v_{n-1}})g(e^{u_0+u_{n-1}})\\&\quad-K(u_--u_{n-1})\\ \geq&-\lambda e^{v_0+v_{n-1}}H(e^{v_0+v_{n-1}})\big(g(e^{u_0+u_-})-g(e^{u_0+u_{n-1}})\big)-K(u_--u_{n-1})\\ =&\lambda e^{v_0+v_{n-1}}H(e^{v_0+v_{n-1}})e^{\xi}G(e^{\xi})(u_--u_{n-1})-K(u_--u_{n-1})\\ \geq&(\lambda H(1)G(1)-K)(u_--u_{n-1})\\ >&0,\end{aligned}$$

where $u_-+u_0\leq\xi\leq u_{n-1}+u_0$. By the maximum principle, we have $u_-\leq u_n$. Using the same argument as before, we get $u_-<u_n$. Similarly, $v_-<v_n$.

It is easy to see that if the system (2.2) has a lower solution, then it admits a solution $(u_\lambda,v_\lambda)=\lim_{n\to\infty}(u_n,v_n)$. If (u'_λ,v'_λ) is any other solution, noting that (u'_λ,v'_λ) is also a lower solution of (2.2), there holds $u_\lambda\geq u'_\lambda$, $v_\lambda\geq v'_\lambda$. Furthermore, proceeding analogously as before, we get

$$(\Delta-K)(u'_\lambda-u_\lambda)\geq(\lambda H(1)G(1)-K)(u'_\lambda-u_\lambda)\geq 0.$$

Assuming that $\max_{x\in V}(u'_\lambda-u_\lambda)(x)=(u'_\lambda-u_\lambda)(x_0)=0$ for some $x_0\in V$, then we conclude that $\Delta(u'_\lambda-u_\lambda)(x_0)\geq 0$. Hence $(u'_\lambda-u_\lambda)(x)=0$ if $x\sim x_0$. The connectedness of G leads to $(u'_\lambda-u_\lambda)(x)\equiv 0$. Similarly, $v'_\lambda(x)\equiv v_\lambda(x)$. This contradicts the assumption $(u_\lambda,v_\lambda)\neq(u'_\lambda,v'_\lambda)$. Therefore, $u_\lambda>u'_\lambda$, $v_\lambda>v'_\lambda$. Thus, in this sense, (u_λ,v_λ) is a unique maximal solution. □

Lemma 2.2. *The system* (2.2) *has a solution if* λ *is big enough.*

Proof. Observe that the functions u_0 and v_0 are bounded since G is finite. Thus, there exists (c_1,c_2) such that $u_0-c_1<0$ and $v_0-c_2<0$. Let $(u_-,v_-)=(-c_1,-c_2)$. It is obvious that

$$\begin{cases}\Delta u_-\geq-\lambda e^{v_0+v_-}H(e^{v_0+v_-})g(e^{u_0+u_-})+\dfrac{4\pi N_1}{|V|},\\ \Delta v_-\geq-\lambda e^{u_0+u_-}G(e^{u_0+u_-})h(e^{v_0+v_-})+\dfrac{4\pi N_2}{|V|},\end{cases} \tag{2.9}$$

if λ is big enough. Hence (u_-,v_-) is a lower solution of the system (2.2). This guarantees the existence of the solution. □

Lemma 2.3. *There exists* $\lambda_c>0$ *such that if* $\lambda>\lambda_c$*, the system* (2.2) *admits a solution, while if* $\lambda<\lambda_c$*, the system* (2.2) *admits no solution.*

Proof. If the system (2.2) admits a solution (u,v), then by integrating both sides of equations in (2.2) on V, we get the necessary condition

$$\lambda\geq\frac{4\pi\max\{N_1,N_2\}}{G(1)H(1)|V|}. \tag{2.10}$$

Define the set

$$\Lambda:=\Big\{\lambda>0\,|\ \lambda \text{ is such that the system (2.2) has a solution}\Big\}.$$

Assume that $\lambda\in\Lambda$ and denote by (u_λ,v_λ) the solution to the system (2.2). For $\lambda_1\in\Lambda$ and $\lambda_1<\lambda_2$, it follows from (2.2) that $(u_{\lambda_1},v_{\lambda_1})$ is a lower solution for (2.2) with $\lambda=\lambda_2$. Hence, we infer that $[\lambda_1,+\infty)\subset\Lambda$ and Λ is an interval. Denote $\lambda_c=\inf\{\lambda\,|\,\lambda\in\Lambda\}$. The inequality (2.10) yields $\lambda_c\geq\frac{4\pi\max\{N_1,N_2\}}{G(1)H(1)|V|}$. This completes the proof. □

Lemmas 2.1 and 2.3 indicate that if $\lambda>\lambda_c$, the system (2.2) has a maximal solution. Denote by $\{(u_\lambda,v_\lambda)\,|\,\lambda>\lambda_c\}$ the family of maximal solutions of (2.2). Assume $\lambda_1>\lambda_2>\lambda_c$. It is easy to check that

$$\begin{aligned}\Delta u_{\lambda_2}&=-\lambda_2\mathrm{e}^{v_0+v_{\lambda_2}}H(\mathrm{e}^{v_0+v_{\lambda_2}})g(\mathrm{e}^{u_0+u_{\lambda_2}})+\frac{4\pi N_1}{|V|}\\&=-\lambda_1\mathrm{e}^{v_0+v_{\lambda_2}}H(\mathrm{e}^{v_0+v_{\lambda_2}})g(\mathrm{e}^{u_0+u_{\lambda_2}})+\frac{4\pi N_1}{|V|}\\&\qquad+(\lambda_1-\lambda_2)\mathrm{e}^{v_0+v_{\lambda_2}}H(\mathrm{e}^{v_0+v_{\lambda_2}})g(\mathrm{e}^{u_0+u_{\lambda_2}})\\&\geq-\lambda_1\mathrm{e}^{v_0+v_{\lambda_2}}H(\mathrm{e}^{v_0+v_{\lambda_2}})g(\mathrm{e}^{u_0+u_{\lambda_2}})+\frac{4\pi N_1}{|V|}.\end{aligned}$$

Similarly,

$$\Delta v_{\lambda_2}\geq-\lambda_1\mathrm{e}^{u_0+u_{\lambda_2}}G(\mathrm{e}^{u_0+u_{\lambda_2}})h(\mathrm{e}^{v_0+v_{\lambda_2}})+\frac{4\pi N_2}{|V|}.$$

Hence, $(u_{\lambda_2},v_{\lambda_2})$ is a lower solution of (2.2) with $\lambda=\lambda_1$. Thus, $u_{\lambda_1}\geq u_{\lambda_2}$and $v_{\lambda_1}\geq v_{\lambda_2}$ by Lemma 2.1. Furthermore, the same argument as before leads to the inequality

$$\Delta(u_{\lambda_2}-u_{\lambda_1})>\lambda_1G(1)H(1)(u_{\lambda_2}-u_{\lambda_1}).$$

Assuming that $\max_{x\in V}(u_{\lambda_2}-u_{\lambda_1})(x)=(u_{\lambda_2}-u_{\lambda_1})(x_0)=0$ for some $x_0\in V$. It follows that $\Delta(u_{\lambda_2}-u_{\lambda_1})(x_0)>0$, which is impossible. Hence $u_{\lambda_1}(x)>u_{\lambda_2}(x)$ for all $x\in V$. Similarly,

$v_{\lambda_1} > v_{\lambda_2}$. Next we use priori estimates to deal with the critical case. We make the decomposition $u_\lambda = \bar{u}_\lambda + u'_\lambda$, where $\bar{u}_\lambda = \frac{1}{|V|}\int_V u_\lambda d\mu$ and $u'_\lambda = u_\lambda - \bar{u}_\lambda$. By (2.2), we get

$$\begin{aligned}\|\nabla u'_\lambda\|_2^2 &= \lambda \int_V e^{v_0+v_\lambda} H(e^{v_0+v_\lambda}) g(e^{u_0+u_\lambda}) u'_\lambda d\mu \\ &\le \lambda G(1) H(1) \int_V |u'_\lambda| d\mu \le C\lambda |V|^{\frac{1}{2}} \|\nabla u'_\lambda\|_2,\end{aligned}$$

where we have used the Poincaré inequality, i.e., (Lemma 6, [12]). Hence

$$\|\nabla u'_\lambda\|_2 \le C\lambda. \tag{2.11}$$

Noting $u_0 + u_\lambda = u_0 + \bar{u}_\lambda + u'_\lambda < 0$, by integration on V, we get

$$\bar{u}_\lambda < -\frac{1}{|V|} \int_V u_0(x) d\mu. \tag{2.12}$$

By integrating the second equation in (2.2) on V, it yields

$$\lambda \int_V e^{u_0+u_\lambda} d\mu \ge \frac{4\pi N_2}{G(1)H(1)}. \tag{2.13}$$

Using the Trudinger-Moser inequality, i.e., (Lemma 7, [12]), we obtain

$$\begin{aligned}\int_V e^{u_0+u_\lambda} d\mu &= \int_V e^{u_0+\bar{u}_\lambda+u'_\lambda} d\mu \le e^{\bar{u}_\lambda} \max_{x\in V} e^{u_0} \int_V e^{u'_\lambda} d\mu \\ &\le C e^{\bar{u}_\lambda} \int_V e^{\|\nabla u'_\lambda\|_2 \frac{u'_\lambda}{\|\nabla u'_\lambda\|_2}} d\mu \le C e^{\bar{u}_\lambda} \int_V e^{\|\nabla u'_\lambda\|_2^2 + \frac{|u'_\lambda|^2}{4\|\nabla u'_\lambda\|_2^2}} d\mu \\ &\le C e^{\bar{u}_\lambda} e^{\|\nabla u'_\lambda\|_2^2}.\end{aligned} \tag{2.14}$$

Then (2.13) and (2.14) give

$$e^{\bar{u}_\lambda} \ge C\lambda^{-1} e^{-\|\nabla u'_\lambda\|_2^2},$$

which together with (2.11) and (2.12) gives

$$|\bar{u}_\lambda| \le C(1+\lambda+\lambda^2).$$

Furthermore,

$$\|u_\lambda\|_{W^{1,2}(V)} \le C(1+\lambda+\lambda^2). \tag{2.15}$$

Similarly,

$$\|v_\lambda\|_{W^{1,2}(V)} \le C(1+\lambda+\lambda^2). \tag{2.16}$$

Set $\lambda_c < \lambda < \lambda_c + 1$. Noting (2.15) and (2.16) and the fact that the space $W^{1,2}(V)$ is precompact, we conclude $u_\lambda \to u_* \in W^{1,2}(V)$, $v_\lambda \to v_* \in W^{1,2}(V)$, pointwisely, as $\lambda \to \lambda_c$. Hence, we deduce that

$$\Delta u_\lambda \to \Delta u_*, \qquad \Delta v_\lambda \to \Delta v_*,$$

$$\lambda e^{v_0+v_\lambda} H(e^{v_0+v_\lambda}) g(e^{u_0+u_\lambda}) \to \lambda_c e^{v_0+v_*} H(e^{v_0+v_*}) g(e^{u_0+u_*}),$$
$$\lambda e^{u_0+u_\lambda} G(e^{u_0+u_\lambda}) h(e^{v_0+v_\lambda}) \to \lambda_c e^{u_0+u_*} G(e^{u_0+u_*}) h(e^{v_0+v_*}),$$

as $\lambda \to \lambda_c$. Thus, (u_*, v_*) is a solution of (2.2) with $\lambda = \lambda_c$. The following lemma is established.

Lemma 2.4. *If $\lambda = \lambda_c$, then the system (2.2) admits a solution.*

Arguing as in proving that (u_λ, v_λ) is monotone, one can show that $u_\lambda > u_*$ and $v_\lambda > v_*$ if $\lambda > \lambda_c$.

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