JOURNAL OF PARTIAL DIFFERENTIAL EQUATIONS
J. Part. Diff. Eq., Vol. **38**, No. 2, pp. 125-141

doi: 10.4208/jpde.v38.n2.1
June 2025

Existence and Multiplicity of Weak Solutions for a Class of Variable Exponent Elliptic Equations

LIU Jingjing[1] and ZHAO Chunshan[2,*]

[1] *Department of Mathematics and Information Science, Zhengzhou University of Light Industry, Zhengzhou 450002, China;*
[2] *Department of Mathematical Sciences, Georgia Southern University, Statesboro, GA 30460, USA.*

Received 28 September 2022; Accepted 11 February 2023

Abstract. In this paper we will study the existence of one, two and three weak solutions for a class of variable exponent elliptic equations under appropriate growth conditions on the nonlinearity. Our technical approach is based on the existence theorems obtained by G. Bonanno.

AMS Subject Classifications: 35D30, 35J70

Chinese Library Classifications: O175.25

Key Words: Weak solutions; p(x)-Laplacian; weighted variable exponent Sobolev spaces.

1 Introduction

In this paper, we will consider existence and multiplicity of weak solutions of the following variable exponent equation

$$\begin{cases} -\mathrm{div}\mathbf{A}(x,\nabla u)+a(x)|u|^{p(x)-2}u+h(x)|u|^{r(x)-2}u=\lambda f(x,u), & \text{in } \Omega, \\ u=0, \quad \text{on } \partial\Omega, \end{cases} \tag{P_λ}$$

where $\Omega\subset\mathbb{R}^N$ is an open bounded domain with smooth boundary, $\lambda\in\mathbb{R}$ and $\mathbf{A}:\Omega\times\mathbb{R}^N\to\mathbb{R}^N$ admits a potential $\mathscr{A}$, with respect to its second variable ξ. We denote by $v_1(x)\ll v_2(x)$ if

$$\operatorname*{essinf}_{x\in\Omega}[v_2(x)-v_1(x)]>0.$$

Next we will state some hypotheses on (P_λ).

*Corresponding author. *Email addresses:* jingjing830306@163.com. czhao@georgiasouthern.edu.

($\mathcal{A}_1$) *The potential* $\mathscr{A}=\mathscr{A}(x,\xi)$ *is a continuous function in* $\Omega\times\mathbb{R}^N$, *with continuous derivative with respect to* ξ, $\mathbf{A}=\partial_\xi\mathscr{A}(x,\xi)$, *and satisfies*

(i) $\mathscr{A}(x,0)=0$ *and* $\mathscr{A}(x,\xi)=\mathscr{A}(x,-\xi)$ *for all* $(x,\xi)\in\Omega\times\mathbb{R}^N$;

(ii) $\mathscr{A}(x,\cdot)$ *is strictly convex in* $\mathbb{R}^N$ *for all* $x\in\mathbb{R}^N$;

(iii) There exist constants $k_1,k_2>0$ *and an exponent* $p(x)\in\mathcal{P}^{log}(\mathbb{R}^N)$ *such that*

$$k_1|\xi|^{p(x)}\le\mathbf{A}(x,\xi)\cdot\xi,\qquad |\mathbf{A}(x,\xi)|\le k_2|\xi|^{p(x)-1} \tag{1.1}$$

for all $(x,\xi)\in\Omega\times\mathbb{R}^N$, *where* $1\ll p(x)\ll N$, *and* $p(x)\in\mathcal{P}^{log}(\Omega)$ *means that* $p(\cdot)$ *is* log–*Hölder continuous in* Ω, *i.e., there exists a constant* p_∞ *satisfying*

$$|p(x)-p(y)|\le\frac{k_3}{\ln\left(e+\frac{1}{|x-y|}\right)}\qquad\text{for all } x,y\in\Omega,$$

$$|p(x)-p_\infty|\le\frac{k_4}{\ln(e+|x|)}\qquad\text{for all } x\in\Omega.$$

($\mathcal{A}_2$) $\mathscr{A}$ *is uniformly convex, i.e., for any* $\varepsilon\in(0,1)$, *there exists a* $\delta=\delta(\varepsilon)\in(0,1)$ *such that either* $|u-v|\le\varepsilon\max\{|u|,|v|\}$, *or* $\mathscr{A}(x,(\xi+\eta)/2)\le\frac{1}{2}(1-\delta)[\mathscr{A}(x,\xi)+\mathscr{A}(x,\eta)]$ *for any* x, ξ, $\eta\in\Omega$.

($\mathcal{H}_1$) *(i)* $a(x)\in L^\infty(\Omega)$, *and for some constant* $k_5\in(0,1]$ *the coefficient* $a(x)$ *satisfies*

$$a(x)\ge k_5(1+|x|)^{-p(x)}\qquad\text{for all } x\in\Omega; \tag{1.2}$$

(ii) $0<h(x)\in L^1(\Omega)$ *and the exponent* $r(x)$ *is continuous in* Ω.

(f_1) $f:\Omega\times\mathbb{R}\to\mathbb{R}$ *is a Carathéodory function satisfying*

$$|f(x,t)|\le k_6+k_7|t|^{q(x)-1},\qquad\text{for all } (x,t)\in\Omega\times\mathbb{R}, \tag{1.3}$$

here k_6, k_7 *are two positive constants,* $q(x)\in C(\overline{\Omega})$ *and* $q(x)\ll p(x)$, $q(x)\ll r(x)$.

By condition ($\mathcal{A}_1$), we have

$$\frac{k_1}{p(x)}|\xi|^{p(x)}\le\mathscr{A}(x,\xi)\le\mathbf{A}(x,\xi)\cdot\xi\le k_2|\xi|^{p(x)}. \tag{1.4}$$

A typical example of elliptic operator $\mathbf{A}$ is $\mathbf{A}(x,\nabla u)=|\nabla u|^{p(x)-2}\nabla u$, so that

$$-\mathrm{div}\mathbf{A}(x,\nabla u)=-\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u)=-\triangle_{p(x)}u,$$

which is called the $p(x)$-Laplacian and satisfies ($\mathcal{A}_1$) and ($\mathcal{A}_2$) if $p\in\mathcal{P}^{\log}(\mathbb{R}^N)$ and $1<p^-\le p^+<N$. Here

$$p^+=\operatorname*{ess\,sup}_{x\in\Omega}p(x),\qquad p^-=\operatorname*{ess\,inf}_{x\in\Omega}p(x).$$

Eq.(P_λ) has been studied in many papers, for example [1–10]. They proved existence, nonexistence and multiplicity results in bounded domain [1] and unbounded domains [9] respectively. The study of the existence of entire solutions of (P_λ) with $f(x,u)=$