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Existence of Nontrivial Steady State of a Chemotaxis Fluid Coupled System

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Abstract. This paper studied a chemotaxis fluid coupled system which models the so-called "chemotactic Boycott effect". Under certain inhomogeneous boundary conditions, we proved the existence of nontrivial steady state which depends only on one variable of such system in $[0,1]^3$. A positive lower bound of the oxygen concentration c was also obtained.

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1 Introduction and main results

In this paper, we investigate the steady state of the following coupled chemotaxis fluid model which was firstly proposed in [1]:

$$\begin{cases} n_t+\mathbf{u}\cdot\nabla n-\Delta n=-\nabla\cdot(n\chi(c)\nabla c), & \text{in } \mathbb{R}^+\times\Omega,\\ c_t+\mathbf{u}\cdot\nabla c-\Delta c=-n\kappa(c), & \text{in } \mathbb{R}^+\times\Omega,\\ \mathbf{u}_t-\Delta\mathbf{u}+(\mathbf{u}\cdot\nabla)\mathbf{u}+\nabla p=-n\nabla\phi, & \text{in } \mathbb{R}^+\times\Omega,\\ \nabla\cdot\mathbf{u}=0, & \text{in } \mathbb{R}^+\times\Omega, \end{cases} \tag{1.1}$$

where c denotes the oxygen concentration , n denotes the bacteria density, and $\mathbf{u}$ denotes the fluid velocity. $\Omega\subset\mathbb{R}^d$ with $d=2,3$ is the physical domain. This system models the so-called "chemotactic Boycott effect", where $\chi(c)$ represents the chemotactic sensitivity and $\nabla\phi$ models the downwards gravitational effect, $\kappa(c)$ models an inactivity threshold

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of the bacteria due to low oxygen supply. Usually(c.f. [2,3]), $\kappa(c)$ was supposed to be a continuously differentiable, monotonically increasing function with $\kappa(0)=0$.

The above time-dependent system (1.1) has been extensively studied during the past decade. For the existence of local and global weak solution under certain assumptions on χ and κ, we refer to [3–5]. See [2] for the study of the Cauchy problem of this system. The existence of global large-data solutions was studied in [6]. For the study of stabilization of such system we refer to [7]. For a survey and critical analysis of chemotaxis models, we refer to [8]. More related studies can be found in [9–19] and references therein.

However, to our knowledge, there is no result on nontrivial steady state of (1.1). The only related work we found is [20], in which the authors proved the existence of nonconstant solution to

$$\begin{cases} -\Delta n+\nabla\cdot(n\nabla c)=0, & \text{in } \Omega,\\ -\Delta c+nc=0, & \text{in } \Omega,\\ \dfrac{\partial c}{\partial \nu}=(\gamma-c)g, \quad \dfrac{\partial n}{\partial \nu}-n\dfrac{\partial c}{\partial \nu}=0, & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

To fill this gap, we consider the following time independent system:

$$\begin{cases} -\Delta n+\mathbf{u}\cdot\nabla n=-\nabla\cdot(n\chi(c)\nabla c), & \text{in } \Omega,\\ -\Delta c+\mathbf{u}\cdot\nabla c=-n\kappa(c), & \text{in } \Omega,\\ -\Delta \mathbf{u}+(\mathbf{u}\cdot\nabla)\mathbf{u}+\nabla p=-n\nabla\phi, & \text{in } \Omega,\\ \quad\nabla\cdot\mathbf{u}=0, & \text{in } \Omega, \end{cases} \tag{1.3}$$

where $\Omega=[0,1]^3$ denotes the unit cubic domain in $\mathbb{R}^3$.

1.1 The boundary conditions

In previous studies of chemotaxis fluid system, many authors used the following homogeneous boundary conditions

$$\frac{\partial n}{\partial \nu}=\frac{\partial c}{\partial \nu}=0, \quad \mathbf{u}=\mathbf{0} \qquad \text{on } \partial\Omega. \tag{1.4}$$

It is easy to see that with this boundary condition, the only steady state will be trivial state $(n_0,0,\mathbf{0})$ with some constant n_0. In fact, by multiplying both sides of the second equation in (1.3) with c, we get

$$\int_\Omega |\nabla c|^2 \mathrm{d}x=-\int_\Omega n\kappa(c)c\,\mathrm{d}x\le 0,$$

since n,c are nonnegative and $\kappa(c)$ are always supposed to be increasing with $\kappa(0)=0$. Thus $c=$constant. Substituting this fact into the first equation in (1.3), and multiplying both sides of this equation with n, we get

$$\int_\Omega |\nabla n|^2 \mathrm{d}x=0.$$