JOURNAL OF PARTIAL DIFFERENTIAL EQUATIONS
J. Part. Diff. Eq., Vol. **38**, No. 2, pp. 171-184

doi: 10.4208/jpde.v38.n2.4
June 2025

Positive Ground State Solutions for a Schrödinger-Newton System with Negative Critical Nonlocal Term

PU Yang[1], ZHU Lijun[2] and LIAO Jiafeng[2,3,*]

[1] *School of Mathematics, Southwestern University of Finance and Economics, Chengdu 611130, China;*
[2] *School of Mathematics and Information, China West Normal University, Nanchong 637009, China;*
[3] *College of Mathematics Education, China West Normal University, Nanchong 637009, China.*

Received 30 December 2023; Accepted 6 May 2024

Abstract. We consider the following Schrödinger-Newton system with negative critical nonlocal term

$$\begin{cases} -\Delta u-\phi|u|^3u=a(x)f(u), & \text{in } \mathbb{R}^3, \\ -\Delta\phi=|u|^5, & \text{in } \mathbb{R}^3, \end{cases}$$

where a and f satisfy some certain conditions. By using the variational method and analytical techniques, we obtain the existence of positive ground state solutions which improves the recent results in the literature.

AMS Subject Classifications: 35B33, 35Q55, 35J60

Chinese Library Classifications: O177.91

Key Words: Schrödinger-Newton system; critical nonlocal term; variational method; ground state solution.

---

1 Introduction

In this paper, we study the existence of positive ground state solutions for the following Schrödinger-Newton system with negative critical nonlocal term

$$\begin{cases} -\Delta u-\phi|u|^3u=a(x)f(u), & \text{in } \mathbb{R}^3, \\ -\Delta\phi=|u|^5, & \text{in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

*Corresponding author. *E-mail address*: liaojiafeng@163.com(J. F. Liao).

where a and f satisfy the following assumptions:
(A) $a\in L^{\frac{3}{2}}(\mathbb{R}^3)\bigcap L^\infty(\mathbb{R}^3)$, $a(x)\geq 0$ and $a(x)\not\equiv 0$;
(F_1) $f\in C(\mathbb{R},\mathbb{R})$, $f(s)\geq 0$ if $s\geq 0$ and $f(s)=0$ if $s\leq 0$;
(F_2) $f(s)=o(s)$ as $s\to 0^+$;
(F_3) $\lim_{s\to+\infty}\frac{f(s)}{s^5}=0$;
(F_4) there exists a $\theta_0\in(0,S/|a|_{\frac{3}{2}})$ such that

$$\left[\frac{f(\tau)}{\tau^3}-\frac{f(t\tau)}{(t\tau)^3}\right]\mathrm{sign}(1-t)+\theta_0\frac{|1-t^2|}{(t\tau)^2}\geq 0,\qquad \forall t>0,\ \tau>0,$$

where $|a|_{\frac{3}{2}}=\left(\int_{\mathbb{R}^3}|a|^{\frac{3}{2}}\mathrm{d}x\right)^{\frac{2}{3}}$ and S is the best Sobolev constant denoted by

$$S=\inf_{u\in D^{1,2}(\mathbb{R}^3)\backslash\{0\}}\frac{\displaystyle\int_{\mathbb{R}^3}|\nabla u|^2\mathrm{d}x}{\left(\displaystyle\int_{\mathbb{R}^3}|u|^6\mathrm{d}x\right)^{\frac{1}{3}}}.$$

The Schrödinger-Newton system was firstly put up by Perkar to explain the quantum mechanics of a polaron. Then it was developed by Choquard giving a description of an electron trapped in its own hold and by Penrose [1] for discussing the self-gravitating matter. For example, a simple particle of the system with mass m is acquired by coupling together the linear Schrödinger equation of quantum mechanics with Poisson equation mechanics. The form of equation is as follows

$$\begin{cases}-\dfrac{h^2}{2m}\Delta u+V(x)u+Uu=0,\\ -\Delta U+4\pi k|u|^2=0,\end{cases}$$

where u denotes the complex wave function, U is the gravitational potential energy, V is a given potential, h is Planck's constant and $k=Gm^2$, G being Newton's constant.

In the recent years, there are a lot of works dealing with solvability or multiplicity of the Schrödinger-Newton system involving subcritical nonlocal term, we refer the reader to [2–11] and references therein. But to the best of our knowledge, fewer papers are devoted to the system with critical nonlocal term, such as [5,12–17]. More precisely, Azzollini et al. in [12] firstly studied the Schrödinger-Newton system with critical nonlocal term as follows

$$\begin{cases}-\Delta u=\lambda u+q\phi|u|^3u, & x\in B_R,\\ -\Delta\phi=q|u|^5, & x\in B_R,\\ u=\phi=0, & \text{on } \partial B_R,\end{cases}$$

where B_R is a ball in $\mathbb{R}^3$ centered at the origin and with radius R. By using the variational method, they obtained the nonexistence result and found a ground state solution depending on λ.