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The Existence and Convergence of Solutions for the Nonlinear Choquard Equations on Groups of Polynomial Growth

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Abstract. In this paper, we study the nonlinear Choquard equation

$$\Delta^2 u - \Delta u + (1+\lambda a(x))u = (R_\alpha * |u|^p)|u|^{p-2}u$$

on a Cayley graph of a discrete group of polynomial growth with the homogeneous dimension $N \geq 1$, where $\alpha \in (0,N), p > \frac{N+\alpha}{N}, \lambda$ is a positive parameter and R_α stands for the Green's function of the discrete fractional Laplacian, which has no singularity at the origin but has same asymptotics as the Riesz potential at infinity. Under some assumptions on $a(x)$, we establish the existence and asymptotic behavior of ground state solutions for the nonlinear Choquard equation by the method of Nehari manifold.

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1 Introduction

The nonlinear Choquard equation

$$-\Delta u + V(x)u = (I_\alpha * |u|^p)|u|^{p-2}u, \qquad x \in \mathbb{R}^N, \tag{1.1}$$

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where

$$I_\alpha = \frac{\Gamma(\frac{N-\alpha}{2})}{\Gamma(\frac{\alpha}{2})\pi^{\frac{N}{2}}2^\alpha|x|^{N-\alpha}}$$

is the Riesz potential, arises in various fields of mathematical physics, such as the description of the quantum theory of a polaron at rest by S. Pekar [1] and the modelling of an electron trapped in its own hole in the work of P. Choquard in 1976. It was also treated as a certain approximation to Hartree-Fock theory of one-component plasma [2]. Sometimes the Eq. (1.1) was also known as the Schrödinger-Newton equation [3], since the convolution part might be treated as a coupling with a Newton equation.

In the last decades, a great deal of mathematical effort has been devoted to the study of existence, multiplicity and properties of solutions to the nonlinear Choquard equation (1.1). If the Eq. (1.1) is equipped with a positive constant potential V, for $N=3,\alpha=2$ and $p=2$, Lieb [2] proved the existence and uniqueness, up to translations, of the ground state solution by the rearrangement technique. Later Lions [4] showed the existence of radially symmetric solutions by variational methods. For $N\geq 3$, Moroz and Van Schaftingen [5] proved the existence of a positive ground state solution for the optimal parameter range of $\frac{N+\alpha}{N} < p < \frac{N+\alpha}{N-2}$, and showed the regularity, positivity, radial symmetry and decay property at infinity. If the Eq. (1.1) is equipped with the deepening potential well $V(x)=1+\lambda a(x)$, where $a(x)$ satisfies the assumptions first introduced by Bartsch and Wang [6], then Alves et al. [7] proved the existence and multiplicity of multi-bump shaped solutions for large λ. In [8], Lü proved the existence of ground state solutions and the sequence of solutions converges strongly to a ground state solution for the problem in a bounded domain if λ is large enough. For more works about the existence and concentration behaviour of solutions to the Choquard equations, we refer readers to [9–11]. In addition, there are intensive studies of the Choquard equations, see papers [12–16].

Nowadays, people paid attention to the analysis on discrete spaces, especially for the nonlinear elliptic equations, see for examples [17–28]. It is worth noting that Zhang and Zhao [21] proved the existence and concentration behaviour of ground state solutions for a class of semilinear elliptic equations equipped with the potential $V(x)=1+\lambda a(x)$ with $a(x)\to+\infty$ as $d(x,x_0)\to+\infty$. Later, the authors [28] generalized their results to biharmonic equations defined on graphs. As far as we know, there are no such results for the nonlinear Choquard equations on graphs. Hence, in this article, we study a class of Choquard equations on graphs that originate from groups.

A group G and a symmetric finite generating set S determine a Cayley graph (G,S). Let $B_r^S(a)=\{x\in G: d^S(x,a)\leq r\}$ be a closed ball of radius r centered at $a\in G$, where d^S is the natural metric on (G,S). We denote by $|B_r^S(a)|=\sharp B_r^S(a)$ the volume of the set $B_r^S(a)$ and $\beta^S(r)=|B_r^S(e)|$, where e is the unit element of G. Then $\beta^S(r)$ is a growth function, one can see [29–33]. A group G is called polynomial growth if, for any $r\geq 1$, $\beta^S(r)\leq Cr^A$, where S is a generating finite set and A is a positive constant. This definition does not depend on the choice of the generating set S since the metrics d^S and d^{S_1} are equivalent to each other on (G,S) for different sets S and S_1. Moreover, in [34], for any group G of