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Selection of Regularization Parameter in the Ambrosio-Tortorelli Approximation of the Mumford-Shah Functional for Image Segmentation

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Abstract. The Ambrosio-Tortorelli functional is a phase-field approximation of the Mumford-Shah functional that has been widely used for image segmentation. It has the advantages of being easy to implement, maintaining the segmentation ability, and Γ-converging to the Mumford-Shah functional as the regularization parameter goes to zero. However, it has been observed in actual computation that the segmentation ability of the Ambrosio-Tortorelli functional varies significantly with different values of the parameter and it even fails to Γ-converge to the original functional for some cases. In this paper we present an asymptotic analysis on the gradient flow equation of the Ambrosio-Tortorelli functional and show that the functional can have different segmentation behavior for small but finite values of the regularization parameter and eventually loses its segmentation ability as the parameter goes to zero when the input image is treatd as a continuous function. This is consistent with the existing observation as well as the numerical examples presented in this work. A selection strategy for the regularization parameter and a scaling procedure for the solution are devised based on the analysis. Numerical results show that they lead to good segmentation of the Ambrosio-Tortorelli functional for real images.

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1. Introduction

Segmentation for a given image is a process to find the edges of objects and partitions the image into separate parts that are relatively smooth. It has been achieved in mathematics by minimizing functionals and multiple theories have been developed. One of the most commonly used functionals, proposed by Mumford and Shah [28], takes the form

$$E[u,\Gamma] = \frac{\alpha}{2}\int_{\Omega\setminus\Gamma} |\nabla u|^2 dx + \beta H^1(\Gamma) + \frac{\gamma}{2}\int_{\Omega} (u-g)^2 dx, \qquad (1.1)$$

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where Ω is a rectangular domain, α, β, and γ are positive parameters, g is the grey level of the input image, u is the target image, Γ denotes the edges of the objects in the image, and $H^1(\Gamma)$ is the one-dimensional Hausdorff measure. Upon minimization, u is close to g, ∇u is small on $\Omega \setminus \Gamma$, and Γ is as short as possible. An optimal image is thus close to the original one and almost piecewise constant. Moreover, the terms in (1.1) represent different and often conflicting objectives, making its minimization and thus image segmentation an interesting and challenging topic to study.

To avoid mathematical difficulties caused by the $H^1(\Gamma)$ term, De Giorgi et al. [8] propose an alternative functional as

$$F[u] = \frac{\alpha}{2}\int_\Omega |\nabla u|^2 dx + \beta H^1(S_u) + \frac{\gamma}{2}\int_\Omega |u-g|^2 dx, \tag{1.2}$$

where S_u is the jump set of u. They show that (1.2) has minimizers in $SBV(\Omega)$ (the space of special functions of bounded variation) and is equivalent to (1.1) in the sense that if $u \in \Omega$ is a minimizer of (1.2), then $(u, \overline{S_u})$ is a minimizer of (1.1), where $\overline{S_u}$ is the closure of S_u.

Although it is a perfectly fine functional to study in mathematics, (1.2) is not easy to implement in actual computation due to the fact that the jump set of the unknown function and its Hausdorff measure are extremely difficult, if not impossible, to compute. To avoid this difficulty, Ambrosio and Tortorelli [1] propose a regularized version as

$$AT_\epsilon[u,\phi] = \frac{\alpha}{2}\int_\Omega (\phi^2 + k_\epsilon)|\nabla u|^2 dx + \beta\int_\Omega \left(\epsilon|\nabla\phi|^2 + \frac{1}{4\epsilon}(1-\phi)^2\right)dx + \frac{\gamma}{2}\int_\Omega |u-g|^2 dx, \tag{1.3}$$

where $\epsilon > 0$ is the regularization parameter, $k_\epsilon = o(\epsilon)$ is a parameter used to prevent the functional from becoming degenerate, and ϕ is a new unknown variable which ideally is an approximation of the characteristic function for the complement of the jump set of u, i.e.,

$$\phi(x) \approx \chi_u(x) \equiv \begin{cases} 0, & \text{if } x \in S_u, \\ 1, & \text{if } x \notin S_u. \end{cases} \tag{1.4}$$

They show that AT_ϵ has minimizers $u \in SBV(\Omega)$ and $\phi \in L^2(\Omega)$ and Γ-converges to $F(u)$. Γ-convergence, first introduced by Ennio de Giorgi, is a concept that guarantees the minimizer of a regularized functional converges to that of the original functional as the regularization parameter goes to 0.

The first finite element approximation for the functional AT_ϵ is given by Bellettini and Coscia [4]. They seek linear finite element approximations u_h and ϕ_h to minimize

$$\begin{aligned} AT_{\epsilon,h}[u_h,\phi_h] = {} & \frac{\alpha}{2}\int_\Omega (\phi_h^2 + k_\epsilon)|\nabla u_h|^2 dx + \beta\int_\Omega \left(\epsilon|\nabla\phi_h|^2 + \frac{1}{4\epsilon}\pi_h((1-\phi_h)^2)\right)dx \\ & + \frac{\gamma}{2}\int_\Omega \pi_h((u_h - g_\epsilon)^2)dx, \end{aligned} \tag{1.5}$$

where π_h is the linear Lagrange interpolation operator and g_ϵ is a smooth function which converges to g in the L^2 norm as $\epsilon \to 0$. They show that $AT_{\epsilon,h}$ Γ-converges to $F(u)$ when