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Multi-Dimensional Image Recovery via Self-Supervised Nonlinear Transform Based a Three-Directional Tensor Nuclear Norm

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Abstract. Recently, the tensor nuclear norm, based on self-supervised nonlinear transformations, has gained significant attention in multidimensional image restoration. However, its primary concept involves solely nonlinear transformations along the third mode of a three-order tensor, which limits its flexibility in dealing with correlations in various modes of high-dimensional data. This paper makes three main contributions. Firstly, we introduce a novel approach called three-directional self-supervised nonlinear transform tensor nuclear norm (3DSTNN), which takes into account nonlinear transformations in all modes and can better represent the global structure of the tensor. Secondly, we suggest a model for multidimensional picture recovery that minimizes ranks by modeling the underlying tensor data as low-rank components subjected to nonlinear transformations. Thirdly, to solve the suggested model, we create an effective algorithm based on the alternating direction method of multipliers (ADMM). In low-rank tensor approximation for image restoration, our approach performs better than the state-of-the-art, according to extensive experimental results on both synthetic and actual datasets.

AMS subject classifications: 94A08, 15A83, 65F22, 65K10

Key words: Three-dimensional nonlinear transform, self-supervised learning, tensor completion, tensor nuclear norm.

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1. Introduction

The swift proliferation of big data applications on a global scale has positioned visual signals, such as images and videos, as primary conveyors of multimedia data. The quality of digital images assumes a pivotal role in data applications and processing. Nevertheless, the observed image or video is frequently subject to data loss [29, 32, 37, 39] and other distortions during imaging, compression, and transmission, resulting in compromised image quality. Image restoration algorithms, designed to reconstruct the original true image from missing data, play a critical role across diverse applications, including medical image analysis [3, 21], remote sensing imagery [22, 28] and image segmentation [23]. Therefore, image recovery based on missing data holds immense significance in the realm of image processing.

In practical applications, high-dimensional data often exhibits intricate intrinsic structures, such as temporal attributes in video sequences and rich spatial-spectral information in hyperspectral images. Traditional representations relying on vectors and matrices have proven inadequate for capturing the complex multilinear structures inherent in such data. Tensors, as higher-order extensions of these conventional representations, provide a more comprehensive depiction of the intrinsic complexity within high-order data structures. It is noteworthy that high-dimensional data in practical scenarios typically demonstrates global correlations that can be effectively characterized by low or approximately low ranks. During the acquisition of high-dimensional data, elements may be lost, necessitating the recovery of these missing elements using information from known data entries – a challenge known as the low rank tensor completion (LRTC) problem. The LRTC problem finds applications in various real-world scenarios, including color image recovery [2, 44], hyperspectral data recovery [6, 31], high-speed compressed video [1], and magnetic resonance imaging (MRI) images recovery [25, 45]. Given the internal correlations and low-rank structure of multidimensional tensors [7, 34], we can use the rank function to represent their internal redundancy. Thus, the low-rank problem can be resolved using the following formulation:

$$
\begin{aligned}
&\min_{\mathcal{X}} \operatorname{rank}(\mathcal{X}), \\
&\text{s.t.}\ \ \mathcal{X}_\Omega = \mathcal{O}_\Omega,
\end{aligned}
\tag{1.1}
$$

where $\mathcal{X}$ represents the underlying low-rank tensor, $\mathcal{O}$ indicates the observed tensor, and $\mathcal{O}_\Omega$ indicates the projection of $\mathcal{X}$ onto the observed set Ω.

Nevertheless, in contrast to matrices, the rank of a tensor lacks a singular definition, a matter that has become a focal point of recent scholarly investigations. Various decomposition methodologies applied to tensors result in diverse interpretations of tensor rank. For instance, the CP decomposition [18] employs a minimum rank one tensor to represent the target tensor. From CP decomposition, CP rank is the minimum quantity of rank-one components necessary to generate the target tensor. Even if it is NP-hard [10], CP rank has found successful applications in addressing tensor completion challenges [38, 43]. Tucker rank [18] is formulated as a vector encompassing the