

Maximum-Principle-Preserving High-Order Conservative Difference Schemes for Convection-Dominated Diffusion Equations

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Abstract. This paper proposes a high-order maximum-principle-preserving (MPP) conservative scheme for convection-dominated diffusion equations. For high-order spatial discretization, we first use the fifth-order weighted compact nonlinear scheme (WCNS5) for the convection term and the sixth-order central difference scheme for the diffusion term. Owing to the nonphysical oscillations caused by the high-order scheme, we further adopt a parameterized MPP flux limiter by modifying a high-order numerical flux toward a lower-order monotone numerical flux to achieve the maximum principle. Subsequently, the resulting spatial scheme is combined with third-order strong-stability-preserving Runge-Kutta (SSPRK) temporal discretization to solve convection-dominated diffusion problems. Several one-dimension (1D) and two-dimension (2D) numerical experiments show that the proposed scheme maintains up to fifth-order accuracy and strictly preserves the maximum principle. The results indicate the proposed scheme's strong potential for solving convection-dominated diffusion and incompressible flow problems.

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Key words: Maximum-principle-preserving, weighted compact nonlinear schemes, parameterized MPP flux limiter, convection-dominated diffusion equations.

1. Introduction

Convection-dominated diffusion equations are used to describe a wide range of physical phenomena in engineering [1], science [41], and mathematics [25]. Such equations involve the interaction between convection and diffusion, resulting in complex, challenging analytical solutions with potential numerical oscillations. Although

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various numerical methods have been proposed to solve the equations, most have difficulties with convection-dominant problems. As such, developing efficient, accurate numerical schemes has become a major research focus.

Consider the following nonlinear convection-diffusion equations in the 1D case:

$$\begin{cases} u_t + f(u)_x = a(u)_{xx}, \\ u(x, 0) = u_0(x) \end{cases} \quad (1.1)$$

with $a'(u) > 0$ and the corresponding 2D problems

$$\begin{cases} u_t + f(u)_x + g(u)_y = a(u)_{xx} + b(u)_{yy}, \\ u(x, y, 0) = u_0(x, y) \end{cases} \quad (1.2)$$

with $a'(u) > 0$, $b'(u) > 0$. The solution to Eq. (1.1) satisfies the maximum principle, that is,

$$\text{if } u_M = \max_x u_0(x), \quad u_m = \min_x u_0(x), \quad \text{then } u(x, t) \in [u_m, u_M], \quad t > 0. \quad (1.3)$$

Physically speaking, if the property (1.3) is violated, it can result in the occurrence of nonphysical solutions. For example, this may lead to negative saturation, density, etc. Of course, they cannot be negative.

Then, the complication of solving convection-dominated diffusion problems arises from the fact that an irregularity can develop in a short period, even when the initial data are smooth, owing to existing the convection term. This poses a great challenge in designing numerical schemes for solving Eq. (1.1). Thus, we need to consider the robustness and high resolution of the numerical schemes. It is well known that first-order schemes are robust near discontinuities but are also highly dissipative. High-order schemes are less dissipative and have a higher resolution, but they typically generate nonphysical oscillations near discontinuities. Different approaches have been developed to eliminate these nonphysical oscillations while maintaining high accuracy. One popular approach is weighted essentially non-oscillatory (WENO) schemes [4, 17, 20, 24, 30, 33, 44]. Another popular approach is weighted compact nonlinear schemes (WCNS), which were developed by Deng and Zhang [11] and later improved by Deng *et al.* [5, 8, 10]. Moreover, Nonomura *et al.* [27, 29] obtained various high-order versions. Compared with the WENO schemes, the WCNS schemes are easier to implement because they use weighted interpolation on the original variables, have a higher resolution at the same order of accuracy, and have a similar ability to capture strong shock waves [28]. Unfortunately, similarly to the WENO schemes, the WCNS schemes do not naturally satisfy the maximum principle because of their high-order nonlinear Lagrange interpretation [18]. To address this issue, Xu [40] proposed the parameterized maximum-principle-preserving (MPP) flux limiter for 1D scalar hyperbolic conservation laws. Liang *et al.* [23] extended the parameterized MPP flux limiter to 2D scalar hyperbolic conservation laws. In addition, the discontinuous Galerkin (DG) methods [6, 7, 31, 35] are also a highly favored approach.