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A Finite Volume Simple WENO Scheme for Convection-Diffusion Equations

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Abstract. A simple weighted essentially non-oscillatory (SWENO) scheme for solving the convection-diffusion equation is proposed in this paper, where a seventh-order SWENO method and the third-order strong stability preserving (SSP) Runge-Kutta method are adopted for discretizing the space and time, respectively. Then the hyperbolic and diffusive part can achieve the seventh- and sixth-order accuracy, respectively. The proposed method has the following advantages. Firstly, negative linear weights are avoided. Secondly, one reconstruction with one stencil is required for the computation of convective and diffusive fluxes. Finally, the new method does not require the transformation while the diffusion coefficients are degenerate. Numerical examples demonstrate that the new method can achieve sixth-order accuracy in the smooth region and guarantee non-oscillatory properties for the discontinuous problems for one- and two-dimensional cases.

AMS subject classifications: 65M10, 78A48
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1. Introduction

The convection-diffusion equation is widely used in many fields, such as environmental science, energy development, fluid mechanics, and gas dynamics. Owing to its wide range of applications, numerical methods for convection-diffusion equations have attracted much attention. The main challenge in solving the convection-diffusion

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equations, especially the convection-dominated equations, is how to make the discrete schemes to achieve high accuracy and maintain stability. Since there are strong discontinuities in the solution of the convection-dominated problem, we expect to maintain high accuracy in the smooth region and non-oscillatory property at the discontinuities.

In the last decades, a variety of high order methods such as ENO [27, 28], WENO [7, 11, 17, 18], Hermite WENO [15], spectral volume method [29, 30], discontinuous Galerkin method [6, 20, 23, 24], finite element method [5, 31], etc, have been developed in the context of the numerical solutions of the hyperbolic conservation laws. Soon after, they are extended for solving convection-diffusion equations we are interested in. Up to now a number of WENO-type methods are adopted for the convection-diffusion equations [4, 10, 21, 22, 32]. These methods can be classified into the finite volume WENO methods [2, 21, 32] and finite difference WENO methods [8, 12, 16]. A nonconventional high order finite volume WENO scheme is proposed for convection-diffusion equations, which can be proved maximum-principle-satisfying [32]. However, the negative linear weights may be present for some Gaussian points, which may lead to instability without special treatment [25]. In order to circumvent the problem, a finite volume WENO scheme for nonlinear parabolic problems is developed [2], where WENO reconstructions with adaptive order (WENO-AO) is employed for spatial discretizations [1,3]. However, the Kirchhoff transformation is required once the diffusion coefficient is degenerate. In addition, two reconstructions with two different stencils are required for the computations of the advective and diffusive flux, respectively. Thus the computation cost is increased.

To reduce the computation cost, a finite volume simple WENO scheme for convection-diffusion equations is developed in this paper, which is first proposed for hyperbolic conservation laws by Qiu and Zhu [33]. The simple WENO method and third-order SSP Runge-Kutta method [9, 26] are employed for the discretizations in space and time, respectively. The new proposed method has several advantages. Firstly, compared with the conventional WENO scheme [18], the presence of the negative linear weights is avoided. Secondly, the calculation of the advective and diffusive flux only need perform one reconstruction with one stencil, which is different from the work [2] and reduce the computation cost. Thirdly, the transformation is avoided when the diffusion coefficient is degenerate compared to the work [2]. Numerical results indicate that the numerical scheme can achieve the sixth-order accuracy in smooth regions and maintain the non-oscillatory property for discontinuous problems.

The paper is structured as follows. The numerical methods for the one- and two-dimensional convection-diffusion equations are presented in detail in Sections 2 and 3, respectively. In Section 4, a selection of numerical examples is presented to demonstrate the performance of the new method. In Section 5, conclusions are given.

2. One-dimensional case

In this section, we consider the numerical solutions of following model with the periodic boundary condition: