Numer. Math. Theor. Meth. Appl.
doi: 10.4208/nmtma.OA-2024-0010

Vol. **17**, No. 4, pp. 904-932
November 2024

A Class of Efficient Multistep Methods for Forward Backward Stochastic Differential Equations

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Received 6 February 2024; Accepted (in revised version) 12 June 2024

Abstract. A class of efficient multistep methods for forward backward stochastic differential equations (FBSDEs) are proposed and studied in this paper. According to the characteristic of our multistep methods and replacing the corresponding Brownian motion increments with appropriate two-point distributed random variables, we obtain a very efficient algorithm to approximate the conditional expectations involved in the multistep methods. It is thanks to this efficient algorithm that we can obtain a class of efficient fully discrete form of high-order methods for the FBSDEs. Finally, some numerical results are presented to illustrate the efficiency of our multistep methods.

AMS subject classifications: 65C30, 60H10, 65L06
Key words: Forward backward stochastic differential equations, multistep methods, efficient high-order methods, conditional expectations.

1. Introduction

In this paper, we consider the forward backward stochastic differential equation (FBSDE) of the integral form

$$\begin{aligned} X(t) &= X(0) + \int_0^t b\big(s, X(s)\big)\, ds + \int_0^t \sigma\big(s, X(s)\big)\, dW(s), \\ y(t) &= \varphi\big(X(T)\big) + \int_t^T f\big(s, X(s), y(s), z(s)\big)\, ds - \int_t^T z(s)\, dW(s), \quad t \in [0, T], \end{aligned} \tag{1.1}$$

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where T is a given positive number, $X(0) \in \mathbb{R}^q, W(t) = (W^1(t), W^2(t), \cdots, W^m(t))^\top$ is a standard m-dimensional Wiener process supported by a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}\}_{0\leq t\leq T})$, the functions $b : \mathbb{R} \times \mathbb{R}^q \longrightarrow \mathbb{R}^q, \sigma : \mathbb{R} \times \mathbb{R}^q \longrightarrow \mathbb{R}^{q\times m}, f : \mathbb{R} \times \mathbb{R}^q \times \mathbb{R}^d \times \mathbb{R}^{d\times m} \longrightarrow \mathbb{R}^d$ and $\varphi : \mathbb{R}^q \longrightarrow \mathbb{R}^d$. Since the study of the regularity of the solution is not at all the aim of this paper. We assume that the functions b, σ, f, φ satisfy some appropriate assumptions (e.g., Lipschitz condition, smoothness and boundedness conditions, etc.) so that the solution of (1.1) has good regularity (see, e.g., [4, 7, 16, 17]).

The existence of a unique solution of FBSDE (1.1) was first addressed in [15]. Moreover, we know from [16, 17] that

$$y(t) = u\big(t, X(t)\big), \quad z(t) = \nabla u\big(t, X(t)\big)\sigma\big(t, X(t)\big), \quad t \in [0, T], \tag{1.2}$$

where ∇u denotes the gradient of $u(t, x) \in C^{1,2}([0, T] \times \mathbb{R}^q)$ with respect to x, and $u(t, x)$ is the classical solution of the following Cauchy problem:

$$\begin{cases} L^{(0)}u(t, x) = -f\big(t, x, u(t, x), \nabla u(t, x)\sigma(t, x)\big), & x \in \mathbb{R}^q, \quad t \in [0, T), \\ u(T, x) = \varphi(x), & x \in \mathbb{R}^q \end{cases} \tag{1.3}$$

with

$$L^{(0)} = \frac{\partial}{\partial t} + \sum_{i=1}^{q} b_i \frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{q} \nu_{ij}\frac{\partial^2}{\partial x_i \partial x_j}, \quad \nu = \sigma\sigma^\top, \tag{1.4}$$

where b_i denotes the i-th element of b and ν_{ij} is the element in i-th row and j-th column of ν. The smoothness of the solution u depends on the smoothness of functions b, σ, f and φ (see, e.g., [7, 10, 26]). Specifically, if $b, \sigma \in C_b^{k,2k}, f \in C_b^{k,2k,2k,2k}, \varphi \in C_b^{2k+\epsilon}, k \in \mathbb{Z}^+, \epsilon \in (0, 1)$, then we have $u \in C_b^{k,2k}$, where $C_b^{k,2k,2k,2k}$ denotes the set of continuously differentiable functions $\phi(t, x, y, z)$ with uniformly bounded partial derivatives $\partial_t^{l_0}\partial_x^{l_1}\partial_y^{l_2}\partial_z^{l_3}\phi$ for $2l_0 + l_1 + l_2 + l_3 \leq 2k$, $C_b^{k,2k}$ denotes the set of functions $\phi(t, x)$ with uniformly bounded partial derivatives $\partial_t^{l_0}\partial_x^{l_1}\phi$ for $2l_0 + l_1 \leq 2k$, and $C_b^{2k+\epsilon}$ denotes the set of functions $\phi(x)$ such that $\partial_x^l\phi, l \leq 2k$ are uniformly bounded and $\partial_x^{2k}\phi$ is Hölder continuous with index ϵ.

As we all know, FBSDEs play an important role in many practical applications. Unfortunately, it is difficult to obtain the analytical solution to most FBSDEs. Therefore, it is very meaningful to find the approximate solution of FBSDE (1.1) by constructing the numerical methods. Up to now, many works on the numerical methods of FBSDEs have been done. For example, some methods based on the relation between the FBSDEs and the corresponding Cauchy problems can be found in [4, 11–14], and some methods developed directly based on FBSDEs can be found in [3, 18, 19, 23, 25].

The linear multistep methods, as a type of important numerical methods for the ordinary differential equations (ODEs), have been successfully extended to solving the FBSDEs (see, e.g., [1, 5, 6, 20–22, 24, 27, 28]). Typically, a high-order weak convergence method for solving forward stochastic differential equations (SDEs) are required for