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Robust Globally Divergence-Free Weak Galerkin Methods for Stationary Incompressible Convective Brinkman-Forchheimer Equations

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Abstract. This paper develops a class of robust weak Galerkin methods for stationary incompressible convective Brinkman-Forchheimer equations. The methods adopt piecewise polynomials of degrees m $(m \geq 1)$ and $m-1$ respectively for the approximations of velocity and pressure variables inside the elements and piecewise polynomials of degrees k $(k = m-1, m)$, and m respectively for their numerical traces on the interfaces of elements, and are shown to yield globally divergence-free velocity approximation. Existence and uniqueness results for the discrete schemes, as well as optimal a priori error estimates, are established. A convergent linearized iterative algorithm is also presented. Numerical experiments are provided to verify the performance of the proposed methods.

AMS subject classifications: 65M60, 65N30

Key words: Brinkman-Forchheimer equations, weak Galerkin method, divergence-free, error estimate.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ $(n = 2, 3)$ be a Lipschitz polygonal/polyhedral domain. We consider the following stationary incompressible convective Brinkman-Forchheimer model:

$$\begin{cases} -\nu\Delta \boldsymbol{u} + \nabla \cdot (\boldsymbol{u} \otimes \boldsymbol{u}) + \alpha|\boldsymbol{u}|^{r-2}\boldsymbol{u} + \nabla p = \boldsymbol{f} & \text{in } \Omega, \\ \nabla \cdot \boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{0} & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Here $\boldsymbol{u} = (u_1, \cdots, u_n)^\top$ is the velocity vector, p the pressure, $\boldsymbol{f}$ a given forcing function, ν the Brinkman coefficient, $\alpha > 0$ the Forchheimer coefficient, and $2 \leq r < \infty$ when

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$n = 2$ and $2 \leq r \leq 6$ when $n = 3$. The operator $\otimes$ is defined by $\boldsymbol{u} \otimes \boldsymbol{v} = (u_i v_j)_{n\times n}$ for $\boldsymbol{v} = (v_1, \cdots, v_n)^\top$.

The Brinkman-Forchheimer model, which can be viewed as the Navier-Stokes equations with a nonlinear damping term, is used to modelling fast flows in highly porous media [20, 43]. In recent years there have developed many numerical algorithms for Brinkman-Forchheimer equations, such as conforming mixed finite element methods [5, 6, 24, 31, 51], nonconforming mixed finite element methods [33], stabilized mixed methods [27,35], multi-level mixed methods [25,41,58,59], parallel finite element algorithms [48,49]. We refer to [4,7,13,18,23,26,32,39,52,57,61,62] for the study of the properties of weak/strong solutions to the Brinkman-Forchheimer equations.

It is well-known that the divergence constraint $\nabla \cdot \boldsymbol{u} = 0$ corresponds to the conservation of mass for incompressible fluid flows, and that numerical methods with poor conservation usually suffer from instabilities [1,19,28,29,38]. Besides, the numerical schemes with exactly divergence-free velocity approximation may automatically lead to pressure-robustness in the sense that the velocity approximation error is independent of the pressure approximation [19,30,36]. We refer to [8,9,11,15,16,21,37,50,55,60] for some divergence-free finite element methods for the incompressible fluid flows.

In this paper we consider a robust globally divergence-free weak Galerkin (WG) finite element discretization of the Brinkman-Forchheimer model (1.1). The WG framework was first proposed in [44, 45] for second-order elliptic problems. It allows the use of totally discontinuous functions on meshes with arbitrary shape of polygons/polyhedra due to the introduction of weakly defined gradient/divergence operators over functions with discontinuity, and has the local elimination property, i.e. the unknowns defined in the interior of elements can be locally eliminated by using the numerical traces defined on the interfaces of elements. We refer to [8, 9, 12, 15–17, 22, 34, 36, 37, 40, 46, 47, 50, 53, 54, 56, 60] for developments and applications of WG methods for fluid flow problems and some other problems. Particularly, a class of robust globally divergence-free weak Galerkin methods were developed in [8] for Stokes equations, and later were extended to solve incompressible quasi-Newtonian Stokes equations [60], natural convection equations [15, 16] and incompressible Magnetohydrodynamics flow equations [56].

The goal of this contribution is to extend the WG methods of [8] to the discretization of the Brinkman-Forchheimer model. The main features of our WG discretization for the model (1.1) are as follows:

- The discretization scheme is arbitrary order, which adopts piecewise polynomials of degrees m $(m \geq 1)$ and $m - 1$ to approximate the velocity and pressure inside the elements, respectively, and piecewise polynomials of degrees k $(k = m-1, m)$ and m to approximate the traces of velocity and pressure on the interfaces of elements, respectively.

- The scheme yields globally divergence-free velocity approximation, which automatically leads to pressure-robustness.