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New Log Basis Functions with a Scaling Factor and Their Applications

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Abstract. In the paper, we introduce two new classes of orthogonal functions, new log orthogonal functions (NLOFs) and generalized new log orthogonal functions (GNLOFs), which are based on generalized Laguerre polynomials. We construct basis approximations analysis for the new orthogonal functions and apply them to solve several typical differential equations. Our errors analysis and numerical results show that our methods based on the new orthogonal functions are particularly suitable for functions which have weak singularities at one endpoint, and can obtain exponential convergence rate, as opposed to low algebraic rates if usual orthogonal polynomials are used.

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Key words: New log orthogonal function, generalized Laguerre functions, singularity, fractional differential equations.

1. Introduction

In spectral methods, global orthogonal basis polynomials or functions are used to approximate solutions of differential equations. The outstanding advantage of spectral methods are high accuracy, which means that ideal convergence accuracy can be achieved with fewer degrees of freedom. These methods are increasingly being used in many practical problems, such as quantum mechanics, astrophysics, fluid dynamics and other fields. Among the existing work, the corresponding spectral methods over the past few decades can been seen in [7, 9, 10, 12, 13, 15, 16, 21, 27, 30, 31].

Among several versions of spectral methods, it is well-known that the spectral methods can achieve spectral accuracy for smooth solutions [2, 8, 22], while it may not have advantage for problems with non-smooth problems. Moreover, many problems

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encounter weakly singular solutions in practical applications, such as in non-smooth areas, in integral equations with singular part, and in fractional differential equations. One popular effective strategy based on finite differences or finite elements is to employ a local adaptive procedure [17], but this method can not be used in spectral method effectively. Under this circumstance, in order to develop spectral methods for problems with non-smooth solutions, one has to construct suitable basis functions which can effectively simulate the non-smooth solutions. If we know some special functions which solve the singular behavior of the problem [24], a popular strategy is to extend the original bases by adding them, such as generalized finite element method [6], the spectral-tau method [20] and the spectral-Galerkin method [4]. Another effective strategy is to construct special orthogonal functions which are suitable for a certain class of problems with singular behaviors. In addition to classical orthogonal polynomials, one can use suitable mappings to classical orthogonal polynomials to construct orthogonal functions in weighted Sobolev spaces, so called mapped spectral methods [1]. This method may have a good effect these problems with singularity, as the essence of this spectral method is to make points near singularity more clustered through the corresponding mapping. Shen and Wang [23] introduced the mapped Legendre spectral methods by the coordinate transformation with a parameter, and provided precise information on how the mapping parameter affect the accuracy. The work in [5] purposed two classes of orthogonal functions through a suitable log mapping based on Laguerre functions which are capable of resolving problems with weak singularities. Basic approximation results and numerical results for fractional differential equations are developed to show the advantages of these basis functions. There are other related mapped works seen [25, 28, 29].

On the other hand, the relevant studies of mapped spectral methods mentioned above [5] are mostly based on standard Laguerre functions $\mathcal{L}_n^{(\alpha,1)}(y)$, which may not appropriate for problems with variable coefficients. In many practical computations, it is necessary to consider transformations based on generalized Laguerre functions $\mathcal{L}_n^{(\alpha,\beta)}(y), \alpha > -1, \beta > 0$. One obvious advantage is that, the mapped basis functions can gather more points at the singularity points by proper selection of parameter β. This phenomena had been proved theoretically in [11], tested numerically in [11, 14] and references therein.

The main goal of this work is to pursuit two new classes of log orthogonal basis (mimic to the basis setting in [5]) by adding a scaling factor β based on the generalized Laguerre functions. The significance is that under such basis, better results can be obtained by selecting parameters β for differential equations. In distinctive contrast to [5], the two new classes of orthogonal basis herein builds in the parameter β, we outline below some important features:

- Under the parameters β, we construct a new log orthogonal basis $S_n^{(\alpha,\beta,\gamma)}$ from the generalized Laguerre functions (NLOFs), which are capable of resolving problems weak singularities. Some projection results about NLOFs are given to show the accuracy by selecting different β.