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A Third-Order Implicit-Explicit Runge-Kutta Method for Landau-Lifshitz Equation with Arbitrary Damping Parameters

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Abstract. A third-order accurate implicit-explicit Runge-Kutta time marching numerical scheme is proposed and implemented for the Landau-Lifshitz-Gilbert equation, which models magnetization dynamics in ferromagnetic materials, with arbitrary damping parameters. This method has three remarkable advantages: (1) only a linear system with constant coefficients needs to be solved at each Runge-Kutta stage, which greatly reduces the time cost and improves the efficiency; (2) the optimal rate convergence analysis does not impose any restriction on the magnitude of damping parameter, which is consistent with the third-order accuracy in time for 1-D and 3-D numerical examples; (3) its unconditional stability with respect to the damping parameter has been verified by a detailed numerical study. In comparison with many existing methods, the proposed method indicates a better performance on accuracy and efficiency, and thus provides a better option for micromagnetics simulations.

AMS subject classifications: 35K61, 65N06, 65N12

Key words: Landau-Lifshitz equation, implicit-explicit Runge-Kutta time discretization, third-order, linear systems with constant coefficients, arbitrary damping.

1. Introduction

The Landau-Lifshitz (LL) equation has been widely used to describe the evolution of magnetic order (magnetization) in continuum ferromagnetic materials [28, 35], which is a vectorial and non-local nonlinear system with non-convex constraint in a point-wise sense and possible degeneracy. A crucial issue in the LL equation is to design efficient and high-order numerical schemes, and considerable progresses have been made in

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the past few decades, see [5, 18, 27, 34, 51] for reviews and references therein. Explicit algorithms (e.g. [2, 8]) and semi-implicit schemes (e.g. [3, 4, 10, 17, 24, 26, 36, 49]) are very popular since they avoid a complicated nonlinear solver while preserving the numerical stability, in comparison with the fully implicit ones (e.g. [7, 25]).

One typical semi-implicit method is based on the backward differentiation formula (BDF) temporal discretization, combined with one-sided extrapolation for nonlinear terms [1, 13, 50]. In [13], the second-order BDF approximation is applied to obtain an intermediate magnetization, and the right-hand-side nonlinear terms are treated in a semi-implicit style with a second-order extrapolation applied to the explicit coefficients. A projection step is further used to preserve the unit length of magnetization at each time step, which poses a non-convex constraint. Such a numerical algorithm, called semi-implicit projection method (SIPM), leads to a linear system of equations with variable coefficients and non-symmetric structure. As a result, no fast solver is available for this numerical system. Meanwhile, an unconditionally unique solvability of the semi-implicit scheme with large damping (SIPM with large damping) has been proved in [12]. The improvement is based on an implicit treatment of the constant-coefficient diffusion term, combined with a fully explicit extrapolation approximation of the nonlinear terms, including the gyromagnetic term and the nonlinear part of the harmonic mapping flow. A direct advantage could be observed in the fact that, the resulting numerical scheme only requires a standard Poisson solver at each time step, which greatly improves the computational efficiency. However, an unconditionally stability is only available for large damping parameter $\alpha > 1$, while most magnetic material models correspond to a parameter $\alpha \ll 1$. In addition, higher-order BDF methods could be applied, while only the first-order and second-order BDF algorithms are unconditionally stable. As analyzed in [1], for the BDF schemes of orders 3 to 5, combined with finite element spatial discretization, the numerical stability requires the damping parameter to be above a positive threshold: $\alpha > \alpha_k$ with $\alpha_k = 0.0913, 0.4041, 4.4348$ for order $k = 3, 4, 5$ respectively. Therefore, it would be highly desirable to design an efficient and higher accurate scheme with no requirement on the damping parameter.

For time-dependent nonlinear partial differential equations in general, implicit-explicit (IMEX) schemes have been extensively used [10]. For the LL equation, the second-order IMEX has been studied in [50]. Two linear systems, with variable coefficients and non-symmetric structure, need to be solved. Hence, IMEX2 can hardly compete with BDF2 in terms of accuracy and efficiency. In a recent work [47], the authors introduce an artificial linear diffusion term and treat it implicitly, while all the remaining terms are treated explicitly. Afterwards, the second-order and the third-order implicit-explicit Runge-Kutta (IMEX-RK2, IMEX-RK3) methods, in which the popular coefficients are derived by the work [6], were proposed for the LL equation in a recent work [32]. Moreover, extensive numerical results have demonstrated that the IMEX-RK2 method has a better performance over the BDF2 approach, in terms of accuracy and efficiency. These IMEX-RK methods worked well for arbitrary damping, and this is a very significant fact in scientific computing, since the damping parameter may be small in most magnetic materials [11]. However, the corresponding theoretical