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A Lagrangian Discontinuous Galerkin Scheme for the Compressible Euler Equations on Unstructured Triangular Meshes

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Abstract. The numerical simulation of fluid dynamics problems plays an important role in the studies of laser inertial confinement fusion (ICF) and magnetic confinement fusion (MCF). Due to the complexity of physical processes and the large deformations of flow field, the numerical simulation of these problems has considerable difficulty. Taking the advantages of the discontinuous Galerkin (DG) method and the Lagrangian scheme, a second-order Lagrangian type scheme for solving the compressible Euler equations is developed on unstructured triangular meshes and implemented by the Runge-Kutta (RK) DG method in this paper. The solver of node velocity in the scheme has good adaptability for many problems. Without considering the material derivatives of basis functions and the Jacobian matrix associated with the map between Lagrangian space and Eulerian space, our scheme is relatively succinct. A HWENO reconstruction is used to eliminate the false oscillations whose stencils involve only the von Neumann neighborhood so that the scheme keeps compact within the DG method. Finally, some numerical examples are presented to illustrate the accuracy, resolution, and robustness of our scheme.

AMS subject classifications: 65M60, 35Q31, 76M10

Key words: Compressible Euler equations, RK-DG method, Lagrangian scheme, unstructured triangular meshes.

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1. Introduction

In the ICF and MCF, etc, there are two typical numerical simulation methods for fluid flows. The first one is the Eulerian method in which the mesh is treated as a fixed reference frame. This method has strong robustness and can be used to deal with the cases involving large deformations. But it is necessary to add other techniques (such as level-set method and ghost fluid method) for capturing the physical interfaces in the simulation of multi-material flows. In addition, it is hard to get the precise physical interfaces, due to the errors from transport calculation. The second one is the Lagrangian method in which the mesh is embedded in the fluid and moves with it. It can catch the material interfaces clearly by using the mesh boundary. So, many scholars have contributed to the development of the Lagrangian method over the past decades.

The Lagrangian method has been proposed for the first time by von Neumann and Richtmyer [38], and then it has been extended to the two-dimensional case later by Wilkins [39]. Considering the difference of the freedom location, the Lagrangian methods can be separated into two kinds: one is the staggered mesh method, in which the velocity is defined at the nodes while the other variables (density, pressure and specific internal energy) are defined inside the cell; the other is the cell-centered mesh method in which all solution variables are defined in the cell center. The development of the two kinds of Lagrangian schemes can been found in [5, 6, 38]. The staggered Lagrangian schemes have been widely applied in fluid dynamics simulations. But it is difficult to construct consistent high-order schemes for all these variables because of the different control volumes used for solution variables in the schemes. The cell-centered Lagrangian schemes can avoid these disadvantages because of the only one control volume used for all solution variables.

The first cell-centered Lagrangian scheme was constructed by Richtmyer [32] with the finite difference method. Then, Dukowicz *et al.* [12, 13] extended it to the multi-dimensional case. The key difficulty of the cell-centered Lagrangian schemes is how to obtain the fluid velocity at the vertex accurately. Maire *et al.* [28] developed a high-order Lagrangian cell-centered finite volume scheme, and a nodal solver was used to compute the vertex velocities and numerical fluxes through the cell interfaces consistently. With the nodal solver in [28], Boscher *et al.* [2] developed a second-order cell-centered pure Lagrangian scheme on unstructured meshes, Li *et al.* [16] presented a cell-centered discretization written in the semi-Lagrangian form by discontinuous Galerkin (DG) method on the quadrilateral meshes. In addition, Xu *et al.* [40] established a control volume staggered Lagrangian scheme in r-z coordinate with the symmetry-preserving property.

The DG method was first introduced by Reed and Hill for the neutron transport equations [31]. Then the RK-DG method was developed by Cockburn and Shu for general nonlinear systems of hyperbolic conservation laws in a series of papers [8–11]. The DG method can deal with the problems including complex geometries and discontinuities flexibly. Its numerical solution is approximated by the polynomials within each volume, hence leading to a natural piecewise high order data representation. In