

A Stabilized Physics Informed Neural Networks Method for Wave Equations

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Abstract. In this article, we propose a novel stabilized physics informed neural networks method (SPINNs) for solving wave equations. In general, this method not only demonstrates theoretical convergence but also exhibits higher efficiency compared to the original PINNs. By replacing the L^2 norm with H^1 norm in the learning of initial condition and boundary condition, we theoretically proved that the error of solution can be upper bounded by the risk in SPINNs. Based on this, we decompose the error of SPINNs into approximation error, statistical error and optimization error. Furthermore, by applying the approximating theory of $ReLU^3$ networks and the learning theory on Rademacher complexity, covering number and pseudo-dimension of neural networks, we present a systematical non-asymptotic convergence analysis on our method, which shows that the error of SPINNs can be well controlled if the number of training samples, depth and width of the deep neural networks have been appropriately chosen. Two illustrative numerical examples on 1-dimensional and 2-dimensional wave equations demonstrate that SPINNs can achieve a faster and better convergence than classical PINNs method.

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1. Introduction

During the past few decades, numerical methods of partial differential equations (PDEs) have been widely studied and applied in various fields of scientific computation [4, 5, 17, 21]. Among these, due to the central significance in solid mechanics, acoustics and electromagnetism, the numerical solution for wave equation attracts considerable attention, and a lot of work has been done to analyze the convergence rate, improve the solving efficiency and deal with practical problems such as boundary conditions. For many real problems with complex region, however, designing efficient and accurate algorithms with practical absorbing boundary conditions is still difficult, especially for problems with irregular boundary. Furthermore, in high-dimensional case, many traditional methods may become even intractable due to the curse of dimensionality, which leads to an exponential increase in degree of freedom with the dimension of problem.

More recently, inspired by the great success of neural network in field of computational science, solving PDEs with deep learning has become as a highly promising topic [1, 3, 8, 13, 14, 19, 28]. Several numerical schemes have been proposed to solve PDEs using neural networks, including the physics-informed neural networks (PINNs) [18], deep Ritz method (DRM) [26] and weak adversarial neural networks (WANs) [27]. While the DRM and WANs formulate the PDEs as variational problems, PINNs solve PDEs by minimizing the residual term penalized with certain boundary and initial constraints. Due to its simplicity and flexibility in its formulation, PINNs have garnered significant attention and have been extended to address a wide range of problems, including the conservation laws [9], the nonlocal Laplacian operator [15] and fractional PDEs [16]. In the field of wave equations, researchers have also successfully applied PINNs to the modeling of scattered acoustic fields [24], including transcranial ultrasound wave [25] and seismic wave [7]. In these works, all of the authors observed an interesting phenomenon that training PINNs without any boundary constraints may lead to a solution under absorbing boundary condition. In another word, the waves obtained by PINNs without boundary loss will be naturally absorbed at the boundary. This phenomenon, in fact, greatly improves the application value of PINNs in wave simulation, especially for inverse scattering problems. On the other hand, although PINNs have been widely used in the simulation of waves, a rigorous numerical analysis of PINNs for wave equations and more efficient training strategy are still needed.

In this work, we propose the stabilized physics informed neural networks for simulation of waves. By replacing the L^2 norm in initial condition and boundary condition with H^1 norm, we obtain a stable PINNs method, in the sense that the error in solution can be upper bounded by the risk during training. It is worth mentioning that, in 2017 a similar idea called Sobolev training has been proposed to improve the efficiency for regression [6]. Later in [20, 23], the authors generalized this idea to the training of PINNs, with applications to heat equation, Burgers' equation, Fokker-Planck equation and elasto-plasticity models. One main difference between our model and these works