

Unconditional Convergence of Linearized TL1 Difference Methods for a Time-Fractional Coupled Nonlinear Schrödinger System

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Abstract. This paper presents a transformed L1 (TL1) finite difference method for the time-fractional coupled nonlinear Schrödinger system. Unconditionally optimal L^2 error estimates of the fully discrete scheme are obtained. The convergence results indicate that the method has an order of 2 in the spatial direction and an order of $2 - \alpha$ in the temporal direction. The error estimates hold without any spatial-temporal stepsize restriction. Such convergence results are obtained by applying a novel discrete fractional Grönwall inequality and the corresponding Sobolev embedding theorems. Numerical experiments for both two-dimensional and three-dimensional models are carried out to confirm our theoretical findings.

AMS subject classifications: 65L04, 65L06, 65M12

Key words: Time-fractional coupled nonlinear Schrödinger system, transformed L1 schemes, unconditionally optimal error estimate, linearly implicit schemes.

1. Introduction

This paper mainly focuses on constructing and analyzing a transformed L1(TL1) linearized finite difference scheme for solving the time-fractional coupled nonlinear

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Schrödinger (TF-CNLS) system

$$\mathbf{i}\partial_t^\alpha u + \Delta u + f_1(u, v)u = 0, \quad x \in \Omega, \quad 0 < t \leq T, \quad (1.1)$$

$$\mathbf{i}\partial_t^\alpha v + \Delta v + f_2(u, v)v = 0, \quad x \in \Omega, \quad 0 < t \leq T, \quad (1.2)$$

$$u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), \quad x \in \Omega, \quad (1.3)$$

$$u(x, t) = 0, \quad v(x, t) = 0, \quad x \in \partial\Omega, \quad t \in [0, T], \quad (1.4)$$

where $\mathbf{i} = \sqrt{-1}$, $\Omega = [0, L_1] \times \cdots \times [0, L_d] \in R^d$ ($d \geq 1$) is a bounded convex and smooth polygon/polyhedron. $f_1(u, v)$ and $f_2(u, v)$ are nonlinear functions. $u(x, t)$ and $v(x, t)$ are space-time dependent complex valued functions describing envelopes of two polarized optical wave packets, which are defined in $\Omega \times [0, T]$. Here ∂_t^α denotes the Caputo fractional time derivative,

$$\partial_t^\alpha u(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x, z)}{\partial z} \frac{1}{(t-z)^\alpha} dz, \quad 0 < \alpha < 1$$

with $\Gamma(\cdot)$ being the usual Gamma function. Since this system of fractional partial differential equations can describe nonlocality behaviors exactly, it is usually used to explain many physical phenomena, such as quantum dynamics. It is challenging to adequately capture nonlocality behaviors when modeling within the framework of classical integer-order calculus [9, 45]. Moreover, it includes some important variants. For instance, when α tends to 1, the system would converge to the classical coupled nonlinear Schrödinger (CNLS) equations. The TF-CNLS equations have numerous significant industrial applications, including beam propagation in Kerr-like photorefractive media, multi-component Bose-Einstein condensates, rogue waves on a multi-soliton background, and description of coupled modes in birefringent media or pulse propagation in multimode fibers [10, 12, 20, 34, 38, 51].

Generally speaking, the solutions of the time-fractional problems have certain initial layers. To describe the initial singularities, it is typically assumed that problem (1.1)-(1.4) owns unique solutions u, v satisfying [19, 42, 44]

$$\left\| \frac{\partial^l u}{\partial t^l} \right\| \leq C_u(1 + t^{\alpha-l}), \quad l = 1, 2, \quad 0 < t \leq T,$$

$$\left\| \frac{\partial^l v}{\partial t^l} \right\| \leq C_v(1 + t^{\alpha-l}), \quad l = 1, 2, \quad 0 < t \leq T,$$

where C_u and C_v are positive constants independent of t and the domain Ω . Up to now, there are many methods to deal with the singularity of fractional derivatives. For instance, Hou and Xu [14] introduced a new class of generalized fractional Jacobi polynomials, and then formulate Galerkin and Petrov-Galerkin spectral methods. More related works by using spectral methods could be seen in [29, 32]. Besides, Jin [18] showed some high-order BDF convolution quadrature methods and more works we refer readers to [8, 49]. In addition, there are finite difference methods on the non-uniformed meshes [6, 30, 43, 50, 53], $L2 - 1_\sigma$ scheme [1, 56, 57] and the transformed