Numer. Math. Theor. Meth. Appl. Vol. **18**, No. 1, pp. 1-30
doi: 10.4208/nmtma.OA-2024-0095 February 2025

Unconditional Convergence of Linearized TL1 Difference Methods for a Time-Fractional Coupled Nonlinear Schrödinger System

Min Li[1], Dongfang Li[1,2,*], Ju Ming[1] and A. S. Hendy[3,4]

[1] *School of Mathematics and Statistics, Huazhong University of Science and Technology, Wuhan 430074, China*
[2] *Hubei Key Laboratory of Engineering Modeling and Scientific Computing, Huazhong University of Science and Technology, Wuhan 430074, China*
[3] *Department of Computational Mathematics and Computer Science, Institute of Natural Sciences and Mathematics, Ural Federal University, Yekaterinburg 620002, Russia*
[4] *Department of Mathematics, Faculty of Science, Benha University, Benha 13511, Egypt*

Received 23 August 2024; Accepted (in revised version) 25 December 2024

Abstract. This paper presents a transformed L1 (TL1) finite difference method for the time-fractional coupled nonlinear Schrödinger system. Unconditionally optimal L^2 error estimates of the fully discrete scheme are obtained. The convergence results indicate that the method has an order of 2 in the spatial direction and an order of $2-\alpha$ in the temporal direction. The error estimates hold without any spatial-temporal stepsize restriction. Such convergence results are obtained by applying a novel discrete fractional Grönwall inequality and the corresponding Sobolev embedding theorems. Numerical experiments for both two-dimensional and three-dimensional models are carried out to confirm our theoretical findings.

AMS subject classifications: 65L04, 65L06, 65M12

Key words: Time-fractional coupled nonlinear Schrödinger system, transformed L1 schemes, unconditionally optimal error estimate, linearly implicit schemes.

1. Introduction

This paper mainly focuses on constructing and analyzing a transformed L1(TL1) linearized finite difference scheme for solving the time-fractional coupled nonlinear

*Corresponding author. *Email addresses:* minli@hust.edu.cn (M. Li), dfli@hust.edu.cn (D. Li), jming@hust.edu.cn (J. Ming), ahmed.hendy@fsc.bu.edu.eg (A.S. Hendy)

Schrödinger (TF-CNLS) system

$$\mathbf{i}\partial_t^\alpha u + \Delta u + f_1(u,v)u = 0, \qquad x \in \Omega, \quad 0 < t \le T, \tag{1.1}$$
$$\mathbf{i}\partial_t^\alpha v + \Delta v + f_2(u,v)v = 0, \qquad x \in \Omega, \quad 0 < t \le T, \tag{1.2}$$
$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad x \in \Omega, \tag{1.3}$$
$$u(x,t) = 0, \quad v(x,t) = 0, \qquad x \in \partial\Omega, \quad t \in [0,T], \tag{1.4}$$

where $\mathbf{i} = \sqrt{-1}$, $\Omega = [0,L_1] \times \cdots \times [0,L_d] \in R^d$ $(d \ge 1)$ is a bounded convex and smooth polygon/polyhedron. $f_1(u,v)$ and $f_2(u,v)$ are nonlinear functions. $u(x,t)$ and $v(x,t)$ are space-time dependent complex valued functions describing envelopes of two polarized optical wave packets, which are defined in $\Omega \times [0,T]$. Here ∂_t^α denotes the Caputo fractional time derivative,

$$\partial_t^\alpha u(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x,z)}{\partial z} \frac{1}{(t-z)^\alpha} dz, \quad 0 < \alpha < 1$$

with $\Gamma(\cdot)$ being the usual Gamma function. Since this system of fractional partial differential equations can describe nonlocality behaviors exactly, it is usually used to explain many physical phenomena, such as quantum dynamics. It is challenging to adequately capture nonlocality behaviors when modeling within the framework of classical integer-order calculus [9, 45]. Moreover, it includes some important variants. For instance, when α tends to 1, the system would converge to the classical coupled nonlinear Schrödinger (CNLS) equations. The TF-CNLS equations have numerous significant industrial applications, including beam propagation in Kerr-like photorefractive media, multi-component Bose-Einstein condensates, rogue waves on a multi-soliton background, and description of coupled modes in birefringent media or pulse propagation in multimode fibers [10, 12, 20, 34, 38, 51].

Generally speaking, the solutions of the time-fractional problems have certain initial layers. To describe the initial singularities, it is typically assumed that problem (1.1)-(1.4) owns unique solutions u, v satisfying [19, 42, 44]

$$\left\| \frac{\partial^l u}{\partial t^l} \right\| \le C_u(1 + t^{\alpha - l}), \quad l = 1,2, \quad 0 < t \le T,$$
$$\left\| \frac{\partial^l v}{\partial t^l} \right\| \le C_v(1 + t^{\alpha - l}), \quad l = 1,2, \quad 0 < t \le T,$$

where C_u and C_v are positive constants independent of t and the domain Ω. Up to now, there are many methods to deal with the singularity of fractional derivatives. For instance, Hou and Xu [14] introduced a new class of generalized fractional Jacobi polynomials, and then formulate Galerkin and Petrov-Galerkin spectral methods. More related works by using spectral methods could be seen in [29, 32]. Besides, Jin [18] showed some high-order BDF convolution quadrature methods and more works we refer readers to [8, 49]. In addition, there are finite difference methods on the non-uniformed meshes [6, 30, 43, 50, 53], $L2-1_\sigma$ scheme [1, 56, 57] and the transformed

L1 method [25–27]. We also recommend [2, 23, 33, 36, 47, 48, 55] to readers for other efficient works.

In the past several years, many numerical results have been done for the related models. Bhrawy [4] applied a new Jacobi spectral collocation method for solving the TF-CNLS system. Hendy [13] constructed a L1-Galerkin spectral scheme for coupled nonlinear space-time fractional Schrödinger equations. Li [28] employed a fast linearized conservative finite element method for the TF-CNLS equations. Qin [40] proposed an Alikhanov linearized Galerkin finite element method for the TF-NLS equations. More details on the time-fractional partial differential equations can be found in [7, 15–17, 35, 37, 52].

It is noted that the early error estimates for the high-dimensional nonlinear problems are usually obtained under a certain space-time step-restriction of $\tau = \mathcal{O}(h^{d/(2p)})$, where d is the dimension, τ is the temporal step size, h is the spatial mesh size and p denotes the temporal convergence order. Such space-time step-restriction is required in the error analysis but unnecessary in the actual computation, see e.g., [3, 11, 54]. In order to remove the restriction, a temporal-spatial error splitting approach was suggested in [21, 22]. This yields the so-called unconditional error analysis for the high-dimensional nonlinear time-dependent problems. The approach was also widely used to obtain the optimally unconditional error estimates of different schemes for time fractional problems, see e.g., [25–27, 31, 41].

In this paper, we follow the ideas in [24, 39] and introduce the following change of variable:

$$t^\alpha = s,$$

we arrive at an equivalent time re-scaled s-fractional differential equation. Then, the temporal discretization is achieved by using the transformed L1 scheme and the extrapolated methods, taking the initial singularity into account. The fully-discrete scheme is developed, with the spatial discretization done using the central finite difference methods. The scheme is quite effective for different parameters α in both two and three dimensional cases. Then, the optimally unconditional convergence results of the fully-discrete scheme are obtained. The results indicate that the optimal L^2 error estimates hold without any spatial-temporal stepsize restrictions, i.e., $\tau = \mathcal{O}(h^c)$ for a positive constant c.

The proof of unconditional convergence results consist of two parts. Firstly, we show the results hold for the following special case, i.e.,

$$f_1(u,v) = |u|^2 + \beta|v|^2, \quad f_2(u,v) = |v|^2 + \beta|u|^2, \tag{1.5}$$

where β is a given constant denoting the wave-wave interaction coefficient characterizing the cross-phase modulation of wave packets. In such a case, the unconditional convergence analysis is proved rigorously by using the corresponding Sobolev embedding theorems and a new discrete fractional Grönwall inequality. Secondly, we show that the unconditional results hold for the equations with the general nonlinear terms

$f_1(u,v)$ and $f_2(u,v)$. To the end, we introduce the so-called temporal-spatial error splitting approach. We provide a routing way to get the optimal unconditional convergence results of the fully-discrete schemes.

The rest of the paper is organized as follows. In Section 2, we present a transformed L1 implicit scheme and the main results. In Section 3, by introducing some notations and lemmas, we give a complete proof of the main results. In Section 4, we present a routing way to obtain the unconditional convergence results. In Section 5, we present several numerical experiments to confirm our theoretical results. Finally, some conclusions are given in Section 6.

Throughout the paper, we let C be a positive constant that is independent of the mesh sizes and may be different in different places.

2. Derivation of the transformed L1 scheme

In this section, we first transform system (1.1)-(1.4) into an equivalent form by introducing the change of variable $t = s^{1/\alpha}$. Then we apply the classical L1 scheme and the finite difference method to the resulting problem, which gives the transformed L1 scheme.

Inspired by the recent research work in [24,39], we introduce the change of variable $t = s^{1/\alpha}$. Denote

$$\tilde{u}(x,y,s) = u\big(x,y,s^{1/\alpha}\big), \quad \tilde{v}(x,y,s) = v\big(x,y,s^{1/\alpha}\big)$$

and then obtain

$$\begin{aligned}
\partial_t^\alpha u &= \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(r)}{\partial r} \frac{1}{(t-r)^\alpha} dr \\
&= \frac{1}{\Gamma(1-\alpha)} \int_0^s \frac{\partial \tilde{u}(r)}{\partial r} \frac{1}{(s^{1/\alpha} - r^{1/\alpha})^\alpha} dr =: D_s^\alpha \tilde{u}, \\
\partial_t^\alpha v &= \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial v(r)}{\partial r} \frac{1}{(t-r)^\alpha} dr \\
&= \frac{1}{\Gamma(1-\alpha)} \int_0^s \frac{\partial \tilde{v}(r)}{\partial r} \frac{1}{(s^{1/\alpha} - r^{1/\alpha})^\alpha} dr =: D_s^\alpha \tilde{v}.
\end{aligned}$$

The system (1.1)-(1.2) can be transformed into the following equivalent form:

$$\mathbf{i} D_s^\alpha \tilde{u} + \Delta \tilde{u} + f_1(\tilde{u},\tilde{v})\tilde{u} = 0, \quad x \in \Omega, \quad 0 < s \le T^\alpha, \tag{2.1}$$

$$\mathbf{i} D_s^\alpha \tilde{v} + \Delta \tilde{v} + f_2(\tilde{u},\tilde{v})\tilde{v} = 0, \quad x \in \Omega, \quad 0 < s \le T^\alpha. \tag{2.2}$$

Similar to the theoretical analysis in [24,39], it is reasonable to assume problem (2.1)-(2.2) admits unique solutions satisfying

$$\left\| \frac{\partial^l \tilde{u}}{\partial s^l} \right\| \le C^* \big(1 + s^{1/\alpha + 1 - l}\big), \quad l = 1,2, \quad 0 < s \le T^\alpha, \tag{2.3}$$

$$\left\| \frac{\partial^l \tilde{v}}{\partial s^l} \right\| \le C^* \big(1 + s^{1/\alpha + 1 - l}\big), \quad l = 1,2, \quad 0 < s \le T^\alpha, \tag{2.4}$$

where $\|\cdot\|$ denotes the usual L^2-norm and C^* is a positive constant independent of s and domain Ω.

Let $h_i = L_i/M_i,\ i = 1,\dots,d$ be the spatial stepsizes with given positive integers $M_i,\ i = 1,\dots,d$. The spatial mesh is defined as $\Omega_h = \{(x_{1,j_1},\cdots,x_{d,j_d})\ |\ x_{i,j_i} = j_i h_i;\ 0 \le j_i \le M_i,\ i = 1,\dots,d\}$. Let N be a positive integer and $\{s_n := n\tau_s\}_{n=0}^N$ be the uniform partition of the interval $[0, T^\alpha]$ with $\tau_s = T^\alpha/N$. Denote the temporal mesh $\Omega_\tau = \{s_n\,|\,0 \le n \le N\}$. Then we define two grid functions on $\Omega_{h\tau} = \Omega_h \times \Omega_\tau$, that is, $\{\tilde{U}^n_{j_1,\cdots,j_d}, \tilde{V}^n_{j_1,\cdots,j_d}\,|\,0 \le j_r \le M_r, 0 \le n \le N;\ r = 1,\dots,d; d \ge 1\}$. Moreover, we denote

$$
\begin{aligned}
J_h &= \{(j_1,\cdots,j_d)\,|\,0 \le j_i \le M_i,\ i = 1,\dots,d\},\\
J_h^{'} &= \{(j_1,\cdots,j_d)\,|\,1 \le j_i \le M_i - 1,\ i = 1,\dots,d\},\\
J_h^{''} &= \{(j_1,\cdots,j_d)\,|\,0 \le j_i \le M_i - 1,\ i = 1,\dots,d\}.
\end{aligned}
$$

Before presenting the scheme, several definitions and notations are introduced below

$$
\begin{aligned}
&\nabla_h U^n_{\vec{j}} = \left(\frac{1}{h_1}\big[U^n_{j_1+1,\cdots,j_d} - U^n_{j_1,\cdots,j_d}\big],\cdots,\frac{1}{h_d}\big[U^n_{j_1,\cdots,j_d+1} - U^n_{j_1,\cdots,j_d}\big]\right),\\
&\Delta_h U^n_{\vec{j}} = \nabla_h\left(\nabla_h U^n_{j_1-1,\cdots,j_d-1}\right),
\end{aligned}
$$

where $\vec{j} = (j_1,\cdots,j_d)$.

For $U^n, V^n \in \Omega_h$, we define the inner product

$$
\langle U^n, V^n\rangle = h_\Delta \sum_{\vec{j}\in J_h^{'}} U^n_{\vec{j}} \bar{V}^n_{\vec{j}},
$$

and some norms

$$
\|U^n\|^2 = \langle U^n, U^n\rangle, \qquad \|U^n\|_{L^p} = \Big(h_\Delta \sum_{\vec{j}\in J_h^{'}} |U^n_{\vec{j}}|^p\Big)^{1/p},
$$

$$
\|\nabla_h U^n\| = \Big(h_\Delta \sum_{\vec{j}\in J_h^{''}} |\nabla_h U^n_{\vec{j}}|^2\Big)^{1/2}, \quad \|U^n\|_{L^\infty} = \max_{\vec{j}\in J_h^{'}} |U^n_{\vec{j}}|,
$$

where $\bar{V}^n_{\vec{j}}$ denotes the conjugate of $V^n_{\vec{j}}$ and $h_\Delta = h_1\cdots h_d$.

Now, we are ready to construct the fully-discrete numerical scheme for problem (2.1)-(2.2). We apply the transformed L1 numerical scheme to approximate the s-fractional differential operator $D_s^\alpha \tilde{u}$, which yields

$$
\begin{aligned}
D^\alpha_{s_n}\tilde{u} &= \frac{1}{\Gamma(1-\alpha)}\int_0^{s_n} \frac{\partial \tilde{u}(r)}{\partial r}\frac{1}{(s_n^{1/\alpha} - r^{1/\alpha})^\alpha}dr\\
&= \frac{1}{\Gamma(1-\alpha)}\sum_{j=1}^n \frac{\tilde{u}(s_j) - \tilde{u}(s_{j-1})}{\tau_s}\int_{s_{j-1}}^{s_j}\frac{1}{(s_n^{1/\alpha} - r^{1/\alpha})^\alpha}dr + \gamma_1^n\\
&= \sum_{j=1}^n a_{n,n-j}\big[\tilde{u}(s_j) - \tilde{u}(s_{j-1})\big] + \gamma_1^n
\end{aligned}
$$

$$= a_{n,0}\tilde{u}(s_n) + \sum_{j=1}^{n-1}(a_{n,n-j} - a_{n,n-j-1})\tilde{u}(s_j) - a_{n,n-1}\tilde{u}(s_0) + \gamma_1^n, \tag{2.5}$$

a similar estimate for $D_s^\alpha \tilde{v}$ yields

$$D_{s_n}^\alpha \tilde{v} = a_{n,0}\tilde{v}(s_n) + \sum_{j=1}^{n-1}(a_{n,n-j} - a_{n,n-j-1})\tilde{v}(s_j) - a_{n,n-1}\tilde{v}(s_0) + \gamma_2^n \tag{2.6}$$

with γ_1^n, γ_2^n being the temporal truncation errors. The coefficients $a_{n,n-j}$ in (2.5) and (2.6) are defined as

$$\begin{aligned} a_{n,n-j} &= \frac{1}{\tau_s \Gamma(1-\alpha)} \int_{s_{j-1}}^{s_j} \frac{1}{(s_n^{1/\alpha} - r^{1/\alpha})^\alpha} dr \\ &= \frac{\alpha}{\tau_s \Gamma(1-\alpha)} \int_{t_{j-1}}^{t_j} \frac{1}{(t_n - t)^\alpha t^{1-\alpha}} dt, \quad t_j = s_j^{1/\alpha}. \end{aligned}$$

It can be further converted into

$$\begin{aligned} a_{n,n-j} &= \frac{\alpha}{\tau_s \Gamma(1-\alpha)} \int_{t_{j-1}/t_n}^{t_j/t_n} \frac{1}{(1-z)^\alpha z^{1-\alpha}} dz \\ &= \frac{\alpha}{\tau_s \Gamma(1-\alpha)} \big(B(t_j/t_n, \alpha, 1-\alpha) - B(t_{j-1}/t_n, \alpha, 1-\alpha) \big), \end{aligned} \tag{2.7}$$

where $B(z, \alpha, 1-\alpha)$ is the usual incomplete Beta function, i.e.

$$B(z, a, b) = \int_0^z x^{a-1}(1-x)^{b-1} dx.$$

According to (2.5) and (2.6), we can define the following discrete fractional operator:

$$D_\tau^\alpha \xi^n = a_{n,0}\xi^n - \sum_{j=1}^{n-1}(a_{n,n-j-1} - a_{n,n-j})\xi^j - a_{n,n-1}\xi^0, \tag{2.8}$$

where $\{\xi^n\}_{n=0}^N$ is a sequence of functions.

Let (u, v) be the solution of the time-fractional coupled nonlinear Schrödinger system (1.1)-(1.4), we denote

$$\tilde{u}_{\vec{j}}^n = \tilde{u}(x_{\vec{j}}, s_n) = u\big(x_{\vec{j}}, s_n^{1/\alpha}\big), \quad \tilde{v}_{\vec{j}}^n = \tilde{v}(x_{\vec{j}}, s_n) = v\big(x_{\vec{j}}, s_n^{1/\alpha}\big)$$

be the solutions of the transformed equivalent system (2.1)-(2.2). Let $\{\tilde{U}_{\vec{j}}^n, \tilde{V}_{\vec{j}}^n\}$ be numerical approximation to $\{\tilde{u}_{\vec{j}}^n, \tilde{v}_{\vec{j}}^n\}$.

With above definitions and notations, the solution $(\tilde{u}, \tilde{v})$ of the system (2.1)-(2.2) satisfies the finite difference equations

$$\mathbf{i} D_\tau^\alpha \tilde{u}_{\vec{j}}^n + \Delta_h \tilde{u}_{\vec{j}}^n + \hat{G}^n(\tilde{u}_{\vec{j}}, \tilde{v}_{\vec{j}})\tilde{u}_{\vec{j}}^n = P_{\vec{j}}^n + \gamma_{1\vec{j}}^n, \quad \vec{j} \in J_h^{'}, \quad 2 \le n \le N, \tag{2.9}$$

$$\mathbf{i} D_\tau^\alpha \tilde{v}_{\vec{j}}^n + \Delta_h \tilde{v}_{\vec{j}}^n + \hat{G}^n(\tilde{v}_{\vec{j}}, \tilde{u}_{\vec{j}})\tilde{v}_{\vec{j}}^n = Q_{\vec{j}}^n + \gamma_{2\vec{j}}^n, \quad \vec{j} \in J_h^{'}, \quad 2 \le n \le N, \tag{2.10}$$

where

$$\hat{G}^n(\tilde{u},\tilde{v}) = 2\big(|\tilde{u}^{n-1}|^2 + \beta|\tilde{v}^{n-1}|^2\big) - \big(|\tilde{u}^{n-2}|^2 + \beta|\tilde{v}^{n-2}|^2\big) \tag{2.11}$$

is a standard extrapolation approximation with the case (1.5), and the definition of the operator D_τ^α mentioned above is given in (2.8).

At the initial step $n=1$, it holds

$$\mathbf{i}D_\tau^\alpha \tilde{u}_{\vec{j}}^1 + \Delta_h \tilde{u}_{\vec{j}}^1 + \big(|\tilde{u}_{\vec{j}}^0|^2 + \beta|\tilde{v}_{\vec{j}}^0|^2\big)\tilde{u}_{\vec{j}}^1 = P_{\vec{j}}^1 + \gamma_{1\vec{j}}^1, \quad \vec{j} \in J_h^{'}, \tag{2.12}$$

$$\mathbf{i}D_\tau^\alpha \tilde{v}_{\vec{j}}^1 + \Delta_h \tilde{v}_{\vec{j}}^1 + \big(|\tilde{v}_{\vec{j}}^0|^2 + \beta|\tilde{u}_{\vec{j}}^0|^2\big)\tilde{v}_{\vec{j}}^1 = Q_{\vec{j}}^1 + \gamma_{2\vec{j}}^1, \quad \vec{j} \in J_h^{'}, \tag{2.13}$$

where $P_{\vec{j}}^n$, $Q_{\vec{j}}^n$, $\gamma_{1\vec{j}}^n$, $\gamma_{2\vec{j}}^n$ are the truncation errors, which are satisfying

$$\begin{aligned}
\gamma_{1\vec{j}}^n &= \mathbf{i}\big(D_\tau^\alpha \tilde{u}_{\vec{j}}^n - D_s^\alpha \tilde{u}(x_{\vec{j}}, s_n)\big), && 1 \le n \le N,\\
\gamma_{2\vec{j}}^n &= \mathbf{i}\big(D_\tau^\alpha \tilde{v}_{\vec{j}}^n - D_s^\alpha \tilde{v}(x_{\vec{j}}, s_n)\big), && 1 \le n \le N,\\
P_{\vec{j}}^n &= \Delta_h \tilde{u}_{\vec{j}}^n - \Delta \tilde{u}(x_{\vec{j}}, s_n) + \hat{G}^n(\tilde{U},\tilde{V})\tilde{u}_{\vec{j}}^n \\
&\quad - \big(|\tilde{u}(x_{\vec{j}}, s_n)|^2 + \beta|\tilde{v}(x_{\vec{j}}, s_n)|^2\big)\tilde{u}(x_{\vec{j}}, s_n), && 2 \le n \le N,\\
Q_{\vec{j}}^n &= \Delta_h \tilde{v}_{\vec{j}}^n - \Delta \tilde{v}(x_{\vec{j}}, s_n) + \hat{G}^n(\tilde{v},\tilde{u})\tilde{v}_{\vec{j}}^n \\
&\quad - \big(|\tilde{v}(x_{\vec{j}}, s_n)|^2 + \beta|\tilde{u}(x_{\vec{j}}, s_n)|^2\big)\tilde{v}(x_{\vec{j}}, s_n), && 2 \le n \le N,\\
P_{\vec{j}}^1 &= \Delta_h \tilde{u}_{\vec{j}}^1 - \Delta \tilde{u}(x_{\vec{j}}, s_1) + \big(|\tilde{u}_{\vec{j}}^0|^2 + \beta|\tilde{v}_{\vec{j}}^0|^2\big)\tilde{u}_{\vec{j}}^1 \\
&\quad - \big(|\tilde{u}(x_{\vec{j}}, s_1)|^2 + \beta|\tilde{v}(x_{\vec{j}}, s_1)|^2\big)\tilde{u}(x_{\vec{j}}, s_1),\\
Q_{\vec{j}}^1 &= \Delta_h \tilde{v}_{\vec{j}}^1 - \Delta \tilde{v}(x_{\vec{j}}, s_1) + \big(|\tilde{v}_{\vec{j}}^0|^2 + \beta|\tilde{u}_{\vec{j}}^0|^2\big)\tilde{v}_{\vec{j}}^1 \\
&\quad - \big(|\tilde{v}(x_{\vec{j}}, s_1)|^2 + \beta|\tilde{u}(x_{\vec{j}}, s_1)|^2\big)\tilde{v}(x_{\vec{j}}, s_1).
\end{aligned}$$

By Taylor formula, we can easily obtain

$$\|P^n\|^2 \le C(\tau_s^2 + h^2)^2, \quad \|Q^n\|^2 \le C(\tau_s^2 + h^2)^2, \quad 2 \le n \le N, \tag{2.14}$$

$$\|P^1\|^2 \le C(\tau_s + h^2)^2, \quad \|Q^1\|^2 \le C(\tau_s + h^2)^2 \tag{2.15}$$

with C a positive constant that is independent of $h_i,\ i=1,\dots,d,\ (d \ge 1)$ and τ_s, where $h = \max_i h_i$.

Omitting the truncation errors $P_{\vec{j}}^n$, $Q_{\vec{j}}^n$, $\gamma_{1\vec{j}}^n$, $\gamma_{2\vec{j}}^n$ in (2.9)-(2.10) and (2.12)-(2.13), the transformed L1 finite difference scheme for system (2.1)-(2.2) reads

$$\mathbf{i}D_\tau^\alpha \tilde{U}_{\vec{j}}^n + \Delta_h \tilde{U}_{\vec{j}}^n + \hat{G}^n(\tilde{U},\tilde{V})\tilde{U}_{\vec{j}}^n = 0, \qquad \vec{j} \in J_h^{'}, \quad 2 \le n \le N, \tag{2.16}$$

$$\mathbf{i}D_\tau^\alpha \tilde{V}_{\vec{j}}^n + \Delta_h \tilde{V}_{\vec{j}}^n + \hat{G}^n(\tilde{V},\tilde{U})\tilde{V}_{\vec{j}}^n = 0, \qquad \vec{j} \in J_h^{'}, \quad 2 \le n \le N, \tag{2.17}$$

$$\mathbf{i}D_\tau^\alpha \tilde{U}_{\vec{j}}^1 + \Delta_h \tilde{U}_{\vec{j}}^1 + \big(|\tilde{U}_{\vec{j}}^0|^2 + \beta|\tilde{V}_{\vec{j}}^0|^2\big)\tilde{U}_{\vec{j}}^1 = 0, \quad \vec{j} \in J_h^{'}, \tag{2.18}$$

$$\mathbf{i}D_\tau^\alpha \tilde{V}_{\vec{j}}^1 + \Delta_h \tilde{V}_{\vec{j}}^1 + \big(|\tilde{V}_{\vec{j}}^0|^2 + \beta|\tilde{U}_{\vec{j}}^0|^2\big)\tilde{V}_{\vec{j}}^1 = 0, \quad \vec{j} \in J_h^{'}, \tag{2.19}$$

subject to

$$\tilde{U}_{\vec{j}}^n|_{\vec{j}\in\partial J_h} = \tilde{V}_{\vec{j}}^n|_{\vec{j}\in\partial J_h} = 0, \qquad 1 \le n \le N, \tag{2.20}$$

$$\tilde{U}_{\vec{j}}^0 = u_0(x_{\vec{j}}), \quad \tilde{V}_{\vec{j}}^0 = v_0(x_{\vec{j}}), \quad \vec{j} \in J_h. \tag{2.21}$$

As can be seen, the solution $\{\tilde{U}_{\vec{j}}^n, \tilde{V}_{\vec{j}}^n \,|\, \vec{j} \in J_h^{'}, \, 2 \le n \le N\}$ at each time step can be calculated by solving two uncoupled linear systems (2.16) and (2.17), and the solution $\{\tilde{U}_{\vec{j}}^1, \tilde{V}_{\vec{j}}^1 \,|\, \vec{j} \in J_h^{'}\}$ can be obtained by solving the systems (2.18) and (2.19). We present the unconditional convergence results of the proposed scheme (2.16)-(2.21) in the following theorem. The proof will be left in the next section.

Theorem 2.1. *Suppose that system* (2.1)-(2.2) *admits a unique solution* $(\tilde{u}, \tilde{v})$ *satisfying* (2.3)-(2.4). *Then the finite difference scheme defined in* (2.16)-(2.21) *admits a unique solution and*

$$\|e^n\|^2 + \|\eta^n\|^2 \le C_0^* \big(\tau_s^{2-\alpha} + h^2\big)^2, \quad 1 \le n \le N, \tag{2.22}$$

where the error functions are defined by

$$e_{\vec{j}}^n = \tilde{u}_{\vec{j}}^n - \tilde{U}_{\vec{j}}^n, \quad \eta_{\vec{j}}^n = \tilde{v}_{\vec{j}}^n - \tilde{V}_{\vec{j}}^n, \quad \vec{j} \in J_h, \quad 0 \le n \le N, \tag{2.23}$$

and C_0^* *is a positive constant independent of* τ_s *and* h.

Remark 2.1. The error estimate (2.22) for high-dimensional nonlinear TF-CNLS systems holds without certain spatial-temporal stepsize restrictions, e.g., $\tau = \mathcal{O}(h^c)$ for a positive constant c.

3. Proof of the main results

In this section, the unique solvability and unconditional convergence analysis of the proposed scheme are proved.

3.1. Existence and uniqueness of the scheme

To show the existence of the approximations $\{(\tilde{U}^n, \tilde{V}^n) \,|\, 1 \le n \le N\}$ for scheme (2.16)-(2.21), we shall use the following Brouwer-type theorem and some important lemmas below.

Lemma 3.1 ([39, Lemma 3.1])**.** *For* $n \ge 1$*, it holds that*

$$0 \le a_{n,n-1} \le a_{n,n-2} \le \cdots \le a_{n,0}.$$

Lemma 3.2 ([52, Lemma 3.2])**.** *For* $n \ge 2$*, it holds that*

$$a_{n,0} \le \frac{2^{\alpha+1/\alpha-2}\tau_s^{-1}}{\alpha^{1-\alpha}\Gamma(2-\alpha)}.$$

Lemma 3.3 ([5]). *Let $(H, \langle\cdot,\cdot\rangle)$ be a finite dimensional inner product space, $\|\cdot\|$ be the associated norm, and let $g: H \to H$ be continuous. Assume, moreover, that*

$$\exists \vartheta > 0, \quad \forall z \in H, \quad \|z\| = \vartheta, \quad \Re\langle g(z), z\rangle \geq 0.$$

Then, there exists an element $z^* \in H$ such that $g(z^*) = 0$ and $\|z^*\| \leq \vartheta$.

Let

$$\Theta = \big\{ v = (v_1, v_2) = (v_{1\vec{j}}, v_{2\vec{j}}) \,|\, \vec{j} \in J_h \big\}$$

and define

$$\begin{aligned}
&\langle v, v^{'} \rangle = \big\langle (v_1, v_2), (v_1^{'}, v_2^{'}) \big\rangle = \langle v_1, v_1^{'} \rangle + \langle v_2, v_2^{'} \rangle, \\
&\|v\|^2 = \|v_1\|^2 + \|v_2\|^2.
\end{aligned}$$

With the above lemmas and denotations, we can get the unique solvability of the proposed scheme, see Theorem 3.1.

Theorem 3.1. *The scheme* (2.16)-(2.21) *is uniquely solvable.*

Proof. First, we show the unique solvability of the scheme holds for $n = 1$. In fact, the existence and uniqueness of the finite difference solution can be obtained by the decaying mass, which is proved later (in Lemma 3.4).

For the case $2 \leq n \leq N$, we prefer another way to get the results. According to (2.8), the scheme (2.16)-(2.17) can be rewritten as

$$\begin{aligned}
&\mathbf{i} a_{n,0} \tilde{U}_{\vec{j}}^n - \sum_{k=1}^{n-1} \mathbf{i}(a_{n,n-k-1} - a_{n,n-k}) \tilde{U}_{\vec{j}}^k - \mathbf{i} a_{n,n-1} \tilde{U}_{\vec{j}}^0 \\
&\quad + \Delta_h \tilde{U}_{\vec{j}}^n + \hat{G}^n(\tilde{U}, \tilde{V}) \tilde{U}_{\vec{j}}^n = 0, \quad \vec{j} \in J_h, \\
&\mathbf{i} a_{n,0} \tilde{V}_{\vec{j}}^n - \sum_{k=1}^{n-1} \mathbf{i}(a_{n,n-k-1} - a_{n,n-k}) \tilde{V}_{\vec{j}}^k - \mathbf{i} a_{n,n-1} \tilde{V}_{\vec{j}}^0 \\
&\quad + \Delta_h \tilde{V}_{\vec{j}}^n + \hat{G}^n(\tilde{V}, \tilde{U}) \tilde{V}_{\vec{j}}^n = 0, \quad \vec{j} \in J_h.
\end{aligned}$$

We first prove the existence of the numerical solution to the scheme. Let $\omega = (\omega_1, \omega_2) \in \Theta$ and define the map $g = (g_1, g_2)$ on Θ as follows:

$$\begin{aligned}
g_1(\omega) = \omega_1 &- \frac{\mathbf{i}}{a_{n,0}} \Delta_h \omega_1 - \frac{\mathbf{i}}{a_{n,0}} \hat{G}(\tilde{U}, \tilde{V}) \omega_1 \\
&- \sum_{k=1}^{n-1} \frac{a_{n,n-k-1} - a_{n,n-k}}{a_{n,0}} \tilde{U}^k - \frac{a_{n,n-1}}{a_{n,0}} \tilde{U}^0,
\end{aligned} \tag{3.1}$$

$$\begin{aligned}
g_2(\omega) = \omega_2 &- \frac{\mathbf{i}}{a_{n,0}} \Delta_h \omega_2 - \frac{\mathbf{i}}{a_{n,0}} \hat{G}(\tilde{V}, \tilde{U}) \omega_2 \\
&- \sum_{k=1}^{n-1} \frac{a_{n,n-k-1} - a_{n,n-k}}{a_{n,0}} \tilde{V}^k - \frac{a_{n,n-1}}{a_{n,0}} \tilde{V}^0.
\end{aligned} \tag{3.2}$$

Multiplying both sides of Eqs. (3.1) and (3.2) with $\omega_{1\vec{j}}$ and $\omega_{2\vec{j}}$ respectively, summing them over Ω_h, then taking the real part of the results, we obtain

$$
\begin{aligned}
\Re\left\langle g_1(\omega),\omega_1\right\rangle &= \|\omega_1\|^2-\sum_{k=1}^{n-1}\frac{a_{n,n-k-1}-a_{n,n-k}}{a_{n,0}}\Re\big\langle \tilde{U}^k,\omega_1\big\rangle-\frac{a_{n,n-1}}{a_{n,0}}\Re\big\langle \tilde{U}^0,\omega_1\big\rangle,\\
\Re\left\langle g_2(\omega),\omega_2\right\rangle &= \|\omega_2\|^2-\sum_{k=1}^{n-1}\frac{a_{n,n-k-1}-a_{n,n-k}}{a_{n,0}}\Re\big\langle \tilde{V}^k,\omega_2\big\rangle-\frac{a_{n,n-1}}{a_{n,0}}\Re\big\langle \tilde{V}^0,\omega_2\big\rangle.
\end{aligned}
$$

By using Lemma 3.1, we then get

$$
\begin{aligned}
&\Re\left\langle g(\omega),\omega\right\rangle = \Re\left\langle g_1(\omega),\omega_1\right\rangle+\Re\left\langle g_2(\omega),\omega_2\right\rangle\\
\geq{}& \|\omega_1\|^2-\sum_{k=1}^{n-1}\frac{a_{n,n-k-1}-a_{n,n-k}}{a_{n,0}}\|\tilde{U}^k\|\,\|\omega_1\|-\frac{a_{n,n-1}}{a_{n,0}}\|\tilde{U}^0\|\,\|\omega_1\|\\
&\quad+\|\omega_2\|^2-\sum_{k=1}^{n-1}\frac{a_{n,n-k-1}-a_{n,n-k}}{a_{n,0}}\|\tilde{V}^k\|\,\|\omega_2\|-\frac{a_{n,n-1}}{a_{n,0}}\|\tilde{V}^0\|\,\|\omega_2\|\\
\geq{}& \|\omega_1\|^2-\frac{1}{2a_{n,0}}\sum_{k=1}^{n-1}\left(a_{n,n-k-1}-a_{n,n-k}\right)\left(\|\tilde{U}^k\|^2+\|\omega_1\|^2\right)\\
&\quad-\frac{a_{n,n-1}}{2a_{n,0}}\left(\|\tilde{U}^0\|^2+\|\omega_1\|^2\right)+\|\omega_2\|^2\\
&\quad-\frac{1}{2a_{n,0}}\sum_{k=1}^{n-1}\left(a_{n,n-k-1}-a_{n,n-k}\right)\left(\|\tilde{V}^k\|^2+\|\omega_2\|^2\right)\\
&\quad-\frac{a_{n,n-1}}{2a_{n,0}}\left(\|\tilde{V}^0\|^2+\|\omega_2\|^2\right)\\
={}& \frac{1}{2}\left(\|\omega_1\|^2+\|\omega_2\|^2\right)-\frac{1}{2a_{n,0}}\sum_{k=1}^{n-1}(a_{n,n-k-1}-a_{n,n-k})\left(\|\tilde{U}^k\|^2+\|\tilde{V}^k\|^2\right)\\
&\quad-\frac{a_{n,n-1}}{2a_{n,0}}\left(\|\tilde{U}^0\|^2+\|\tilde{V}^0\|^2\right)\\
={}& \frac{1}{2}\left(\|\omega\|^2-\frac{1}{a_{n,0}}\sum_{k=1}^{n-1}\left(a_{n,n-k-1}-a_{n,n-k}\right)\|(\tilde{U}^k,\tilde{V}^k)\|^2-\frac{a_{n,n-1}}{a_{n,0}}\|(\tilde{U}^0,\tilde{V}^0)\|^2\right).
\end{aligned}\tag{3.3}
$$

Taking

$$
\vartheta^2=\frac{1}{a_{n,0}}\sum_{k=1}^{n-1}\left(a_{n,n-k-1}-a_{n,n-k}\right)\|(\tilde{U}^k,\tilde{V}^k)\|^2+\frac{a_{n,n-1}}{a_{n,0}}\|(\tilde{U}^0,\tilde{V}^0)\|^2,
$$

we can get that there exists a $\vartheta>0$ such that for any $\omega\in\Theta$ satisfying $\|\omega\|=\vartheta$, it holds $\Re\langle g(\omega),\omega\rangle\geq 0$ evidently. By virtue of Lemma 3.3, there exists an element $\omega^*\in\Theta$

satisfying $g(w^*) = 0$, and we can also get

$$\|\omega^*\| \le \left(\frac{1}{a_{n,0}} \sum_{k=1}^{n-1} (a_{n,n-k-1} - a_{n,n-k}) \, \|(\tilde{U}^k, \tilde{V}^k)\|^2 + \frac{a_{n,n-1}}{a_{n,0}} \|(\tilde{U}^0, \tilde{V}^0)\|^2 \right)^{1/2}.$$

We then go on to prove the uniqueness of the solution to scheme (3.1) and (3.2). Suppose the scheme admits two solutions $z = (z_1, z_2)$ and $\tilde{z} = (\tilde{z}_1, \tilde{z}_2)$, which satisfy

$$z_1 - \frac{\mathbf{i}}{a_{n,0}} \Delta_h z_1 - \frac{\mathbf{i}}{a_{n,0}} \hat{G}(\tilde{U}, \tilde{V}) z_1 - \sum_{k=1}^{n-1} \frac{a_{n,n-k-1} - a_{n,n-k}}{a_{n,0}} \tilde{U}^k - \frac{a_{n,n-1}}{a_{n,0}} \tilde{U}^0 = 0, \tag{3.4}$$

$$z_2 - \frac{\mathbf{i}}{a_{n,0}} \Delta_h z_2 - \frac{\mathbf{i}}{a_{n,0}} \hat{G}(\tilde{V}, \tilde{U}) z_2 - \sum_{k=1}^{n-1} \frac{a_{n,n-k-1} - a_{n,n-k}}{a_{n,0}} \tilde{V}^k - \frac{a_{n,n-1}}{a_{n,0}} \tilde{V}^0 = 0, \tag{3.5}$$

$$\tilde{z}_1 - \frac{\mathbf{i}}{a_{n,0}} \Delta_h \tilde{z}_1 - \frac{\mathbf{i}}{a_{n,0}} \hat{G}(\tilde{U}, \tilde{V}) \tilde{z}_1 - \sum_{k=1}^{n-1} \frac{a_{n,n-k-1} - a_{n,n-k}}{a_{n,0}} \tilde{U}^k - \frac{a_{n,n-1}}{a_{n,0}} \tilde{U}^0 = 0, \tag{3.6}$$

$$\tilde{z}_2 - \frac{\mathbf{i}}{a_{n,0}} \Delta_h \tilde{z}_2 - \frac{\mathbf{i}}{a_{n,0}} \hat{G}(\tilde{V}, \tilde{U}) \tilde{z}_2 - \sum_{k=1}^{n-1} \frac{a_{n,n-k-1} - a_{n,n-k}}{a_{n,0}} \tilde{V}^k - \frac{a_{n,n-1}}{a_{n,0}} \tilde{V}^0 = 0. \tag{3.7}$$

Letting $\varepsilon = (\varepsilon_1, \varepsilon_2) = z - \tilde{z}$ and subtracting (3.4)-(3.5) from (3.6)-(3.7) respectively, we have

$$\varepsilon_1 - \frac{\mathbf{i}}{a_{n,0}} \Delta_h \varepsilon_1 - \frac{\mathbf{i}}{a_{n,0}} \hat{G}(\tilde{U}, \tilde{V}) \varepsilon_1 = 0, \tag{3.8}$$

$$\varepsilon_2 - \frac{\mathbf{i}}{a_{n,0}} \Delta_h \varepsilon_2 - \frac{\mathbf{i}}{a_{n,0}} \hat{G}(\tilde{V}, \tilde{U}) \varepsilon_2 = 0. \tag{3.9}$$

Computing the inner product of (3.8)-(3.9) with $\varepsilon = (\varepsilon_1, \varepsilon_2)$ and then taking the real part of the result, we obtain

$$\|\varepsilon\|^2 = 0.$$

This implies that $\|\varepsilon\| = 0$. Consequently, the uniqueness of the solution to scheme (2.16)-(2.21) is derived and the proof is complete. $\square$

3.2. Stability and some notations

In this subsection, we present some important lemmas which will be used later.

Lemma 3.4. *Let* $(\tilde{U}^n,\ \tilde{V}^n)$ *be solution of the difference scheme* (2.16)-(2.21). *Then the following inequalities:*

$$\|\tilde{U}^n\| \le \|\tilde{U}^0\|, \quad \|\tilde{V}^n\| \le \|\tilde{V}^0\|, \quad 1 \le n \le N \tag{3.10}$$

hold.

Proof. We first prove it holds for $n=1$. Computing the inner product of (2.18) with $\tilde{U}^1$, and taking the imaginary part of the equation, we obtain

$$\|\tilde{U}^1\|^2 = \Re\langle \tilde{U}^0, \tilde{U}^1\rangle \le \|\tilde{U}^0\|\|\tilde{U}^1\|,$$

which implies that $\|\tilde{U}^1\| \le \|\tilde{U}^0\|$. Similarly, we can prove that $\|\tilde{V}^1\| \le \|\tilde{V}^0\|$.

Multiplying (2.16) by $\tilde{U}^n_{\vec{j}}$ and sum them up for $\vec{j} \in J_h'$ to get

$$\mathbf{i}\langle D_\tau^\alpha \tilde{U}^n, \tilde{U}^n\rangle - \|\nabla_h \tilde{U}^n\|^2 + \sum_{\vec{j}} h_\Delta \hat{G}^n(\tilde{U}, \tilde{V})|\tilde{U}^n_{\vec{j}}|^2 = 0.$$

The imaginary part of the above equation implies

$$\Re\langle D_\tau^\alpha \tilde{U}^n, \tilde{U}^n\rangle = 0,$$

which indicates

$$\begin{aligned}
a_{n,0}\|\tilde{U}^n\|^2 &= \sum_{k=1}^{n-1}(a_{n,n-k-1} - a_{n,n-k})\Re\langle \tilde{U}^k, \tilde{U}^n\rangle + a_{n,n-1}\Re\langle \tilde{U}^0, \tilde{U}^n\rangle \\
&\le \sum_{k=1}^{n-1}\frac{1}{2}(a_{n,n-k-1} - a_{n,n-k})\left(\|\tilde{U}^k\|^2 + \|\tilde{U}^n\|^2\right) + \frac{1}{2}a_{n,n-1}\left(\|\tilde{U}^0\|^2 + \|\tilde{U}^n\|^2\right).
\end{aligned}$$

It can be further implied that

$$a_{n,0}\|\tilde{U}^n\|^2 \le \sum_{k=1}^{n-1}(a_{n,n-k-1} - a_{n,n-k})\|\tilde{U}^k\|^2 + a_{n,n-1}\|\tilde{U}^0\|^2.$$

We prove (3.10) by mathematical induction. Suppose that the inequalities (3.10) hold for $1 \le n \le m-1$, $m \ge 2$. We prove that (3.10) also holds when $n = m$. If $n = m$, we have

$$\begin{aligned}
a_{m,0}\|\tilde{U}^m\|^2 &\le \sum_{k=1}^{m-1}(a_{m,m-k-1} - a_{m,m-k})\max_{1\le k\le m-1}\|\tilde{U}^k\|^2 + a_{m,m-1}\|\tilde{U}^0\|^2 \\
&\le \sum_{k=1}^{m-1}(a_{m,m-k-1} - a_{m,m-k})\|\tilde{U}^0\|^2 + a_{m,m-1}\|\tilde{U}^0\|^2 = a_{m,0}\|\tilde{U}^0\|^2,
\end{aligned}$$

which implies $\|\tilde{U}^n\| \le \|\tilde{U}^0\|$. Similarly, we can prove that $\|\tilde{V}^n\| \le \|\tilde{V}^0\|$ also holds for $1 \le n \le N$. The proof is complete. □

We also need the help of the following lemmas before we give the detailed proof of Theorem 2.1.

Lemma 3.5 ([24, Lemma 3.2]). *Under the assumptions* (2.3)-(2.4)*, we have, for* $n = 1, 2, \ldots, N$*,*

$$\|\gamma_1^n\| \le C\tau_s^{1-\alpha}, \quad \|\gamma_2^n\| \le C\tau_s^{1-\alpha},$$

where γ_1^n *and* γ_2^n *are the temporal truncation errors and are well defined in* (2.5) *and* (2.6).

Lemma 3.6 ([46])**.** *Let $\{\omega_{\vec{j}}\}$ be a mesh function defined in Ω_h for the three-dimensional case. Then*

$$\begin{aligned} &\|\omega\|_{L^p} \le C\|\omega\|_{L^2}^{3/p-1/2}\big\{\|\nabla_h\omega\|_{L^2}+\|\omega\|_{L^2}\big\}^{3/2-3/p}, && 2\le p\le 6,\\ &\|w\|_{L^p} \le Ch^{3/p-3/q}\|w\|_{L^q}, && 1\le q\le p\le\infty, \end{aligned}$$

where C is a constant independent of h and the mesh function ω.

Lemma 3.7. *For any finite time $s_N = T^\alpha > 0$ and a specified nonnegative sequence $\{\lambda_l\}_{l=0}^{N-1}$, assume that there exists a constant λ independent of time step τ_s such that $\lambda \ge \sum_{l=0}^{N-1}\lambda_l$. Suppose that the grid function sequence $\{v^n \mid n\ge 0\}$ satisfies*

$$D_\tau^\alpha(v^n)^2 \le \sum_{l=1}^{n}\lambda_{n-l}(v^l)^2 + \mu v^n(\gamma_1^n+\gamma_2^n+R), \quad n\ge 1, \tag{3.11}$$

where μ is a constant and γ_1^n, γ_2^n are defined in (2.5)-(2.6)*. Then, there are some positive constants C_1^* and τ_s^* such that, when $\tau_s\le\tau_s^*$, it holds that*

$$v^n \le 2E_\alpha(2\lambda s_n)\left(v^0+\mu\left(2C_1^*\tau_s^{2-\alpha}+\frac{s_n}{\Gamma(1+\alpha)}R\right)\right), \quad 1\le n\le N, \tag{3.12}$$

where

$$E_\alpha(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(1+k\alpha)}.$$

Proof. As in [52], the discrete convolution kernel is defined as follows:

$$p_{n,0}=\frac{1}{a_{n,0}}, \quad p_{n,n-j}=\frac{1}{a_{j,0}}\sum_{k=j+1}^{n}(a_{k,k-j-1}-a_{k,k-j})p_{n,n-k}, \quad 1\le j\le n-1.$$

The following properties can be found in [52]:

$$\sum_{j=1}^{n-1}p_{n,n-j}\frac{s_j^m}{\Gamma(1+m\alpha)} \le \frac{s_n^{m+1}}{\Gamma(1+m\alpha+\alpha)}, \quad m=0,1,2,\dots, \tag{3.13}$$

$$\sum_{j=1}^{n}p_{n,n-j}\gamma_1^j \le C_1^*\tau_s^{2-\alpha}. \tag{3.14}$$

We will obtain the global error $\mu\sum_{j=1}^{n}p_{n,n-j}(\gamma_1^j+\gamma_2^j+R)$ by letting $m=0$ in (3.13) and applying (3.14). It holds that

$$\mu\sum_{j=1}^{n}p_{n,n-j}\big(\gamma_1^j+\gamma_2^j+R\big) \le 2C_1^*\mu\tau_s^{2-\alpha}+\frac{\mu s_n}{\Gamma(1+\alpha)}R. \tag{3.15}$$

With the mentioned results, one can get the required results by following the proof of [52, Theorem 3.1]. □

3.3. Proof of Theorem 2.1

In this subsection, a complete proof of Theorem 2.1 will be given. A novel discrete fractional Grönwall inequality is used. This is because the inequality works for the new time discretization, i.e., the so called transformed L1 scheme. The scheme is derived by a smoothing transformation $t = s^{1/\alpha}$, which can be helpful to deal with the initial singular layer.

Proof. We mainly focus on presenting the proof for three-dimensional problem. Noting that the error functions defined in (2.23) and subtracting (2.16)-(2.17) and (2.18)-(2.19) from (2.9)-(2.10) and (2.12)-(2.13), respectively, we get

$$\mathbf{i}D_\tau^\alpha e_{\vec{j}}^n + \Delta_h e_{\vec{j}}^n + \hat{G}^n(\tilde{u},\tilde{v})\tilde{u}_{\vec{j}}^n - \hat{G}^n(\tilde{U},\tilde{V})\tilde{U}_{\vec{j}}^n = P_{\vec{j}}^n + \gamma_{1\vec{j}}^n, \quad \vec{j} \in J_h', \quad 2 \le n \le N, \tag{3.16}$$

$$\mathbf{i}D_\tau^\alpha \eta_{\vec{j}}^n + \Delta_h \eta_{\vec{j}}^n + \hat{G}^n(\tilde{v},\tilde{u})\tilde{v}_{\vec{j}}^n - \hat{G}^n(\tilde{V},\tilde{U})\tilde{V}_{\vec{j}}^n = Q_{\vec{j}}^n + \gamma_{2\vec{j}}^n, \quad \vec{j} \in J_h', \quad 2 \le n \le N, \tag{3.17}$$

$$\mathbf{i}D_\tau^\alpha e_{\vec{j}}^1 + \Delta_h e_{\vec{j}}^1 + \big(|\tilde{u}_{\vec{j}}^0|^2 + \beta|\tilde{v}_{\vec{j}}^0|^2\big)e_{\vec{j}}^1 = P_{\vec{j}}^1 + \gamma_{1\vec{j}}^1, \quad \vec{j} \in J_h', \tag{3.18}$$

$$\mathbf{i}D_\tau^\alpha \eta_{\vec{j}}^1 + \Delta_h \eta_{\vec{j}}^1 + \big(|\tilde{v}_{\vec{j}}^0|^2 + \beta|\tilde{u}_{\vec{j}}^0|^2\big)\eta_{\vec{j}}^1 = Q_{\vec{j}}^1 + \gamma_{2\vec{j}}^1, \quad \vec{j} \in J_h', \tag{3.19}$$

$$e_{\vec{j}}^0 = \eta_{\vec{j}}^0 = 0, \quad \vec{j} \in J_h', \tag{3.20}$$

$$e_{\vec{j}}^n|_{\partial J_h} = \eta_{\vec{j}}^n|_{\partial J_h} = 0, \quad 0 \le n \le N. \tag{3.21}$$

Multiplying both sides of Eq. (3.18) with $e_{\vec{j}}^1$ and summing them over J_h', we have

$$\mathbf{i}a_{1,0}\|e^1\|^2 - \|\nabla_h e^1\|^2 + \sum_{\vec{j}\in J_h'} h_\Delta\big(|\tilde{u}_{\vec{j}}^0|^2 + \beta|\tilde{v}_{\vec{j}}^0|^2\big)|e_{\vec{j}}^1|^2 = \langle P^1 + \gamma_1^1, e^1\rangle,$$

where $h_\Delta = h_1h_2h_3$. The imaginary part of the above equation implies

$$a_{1,0}\|e^1\|^2 = \Im\langle P^1 + \gamma_1^1, e^1\rangle, \tag{3.22}$$

and the real part implies

$$\|\nabla_h e^1\|^2 \le \max_{x\in\Omega}\big\{|u_0(x)|^2 + \beta|v_0(x)|^2\big\}\|e^1\|^2 + \Re\langle P^1 + \gamma_1^1, e^1\rangle. \tag{3.23}$$

By (2.7), we get

$$a_{1,0} = \frac{\alpha}{\tau_s\Gamma(1-\alpha)}B(\alpha, 1-\alpha) = \frac{\Gamma(1+\alpha)}{\tau_s},$$

where $B(\cdot,\cdot)$ represents the Beta function. Then it follows (3.22) and (3.23) that

$$\begin{aligned} &\|e^1\| \le \frac{1}{a_{1,0}}\big(\|P^1\| + \|\gamma_1^1\|\big) \le C(\tau_s^{2-\alpha} + h^2), \\ &\tau_s\|\nabla_h e^1\|^2 \le C\big(\tau_s^{2-\alpha} + h^2\big)^2, \end{aligned} \tag{3.24}$$

where (2.14), (2.15) and Lemma 3.5 have been used.

Similarly, we can prove that

$$\begin{aligned}&\|\eta^1\| \le C\big(\tau_s^{2-\alpha}+h^2\big),\\&\tau_s\|\nabla_h\eta^1\|^2 \le C\big(\tau_s^{2-\alpha}+h^2\big)^2,\end{aligned} \tag{3.25}$$

which implies that (2.22) holds when $n=1$.

Now, we shall prove a slightly stronger inequality holds for $1\le m\le n$. There exist positive constants τ_s^* and h_0 such that, when $\tau_s\le\tau_s^*$, $h\le h_0$, it holds

$$\|e^m\|^2+\|\eta^m\|^2+\tau_s\left(\|\nabla_h e^m\|^2+\|\nabla_h\eta^m\|^2\right)\le K_1\big(\tau_s^{2-\alpha}+h^2\big)^2, \tag{3.26}$$

where K_1 is a positive constant independent of τ_s, h and m. Clearly from (3.24) and (3.25) we can see (3.26) holds for $m=1$. The inequality is proved by mathematical induction. We can assume that (3.26) holds for $2\le m\le n-1$, all we need is to prove that (3.26) also holds when $m=n$. Firstly, multiplying both sides of Eq. (3.16) with $e^n_{\vec{j}}$ and summing them over J_h' to arrive at

$$\mathbf{i}\langle D_\tau^\alpha e^n,e^n\rangle-\|\nabla_h e^n\|^2+R_1^n=\langle P^n+\gamma_1^n,e^n\rangle, \tag{3.27}$$

where

$$R_1^n=h_\Delta\sum_{\vec{j}\in J_h'}\left[\hat{G}^n(\tilde{u},\tilde{v})\tilde{u}^n_{\vec{j}}-\hat{G}^n(\tilde{U},\tilde{V})\tilde{U}^n_{\vec{j}}\right]e^n_{\vec{j}}.$$

Taking the imaginary part of the above equation gives

$$\Re\langle D_\tau^\alpha e^n,e^n\rangle+\Im(R_1^n)=\Im\langle P^n+\gamma_1^n,e^n\rangle. \tag{3.28}$$

We rewrite R_1^n by

$$\begin{aligned}R_1^n&=h_\Delta\sum_{\vec{j}\in J_h'}\left[\big(\hat{G}^n(\tilde{u},\tilde{v})-\hat{G}^n(\tilde{U},\tilde{V})\big)\tilde{U}^n_{\vec{j}}+\hat{G}^n(\tilde{u},\tilde{v})e^n_{\vec{j}}\right]e^n_{\vec{j}}\\&=h_\Delta\sum_{\vec{j}\in J_h'}\hat{G}^n(\tilde{u},\tilde{v})|e^n_{\vec{j}}|^2+h_\Delta\sum_{\vec{j}\in J_h'}\big(\hat{G}^n(\tilde{u},\tilde{v})-\hat{G}^n(\tilde{U},\tilde{V})\big)\big(\tilde{u}^n_{\vec{j}}-e^n_{\vec{j}}\big)e^n_{\vec{j}}\\&=:R_{11}^n+R_{12}^n.\end{aligned} \tag{3.29}$$

By noting (2.11), it is easy to obtain

$$\Im(R_{11}^n)=0, \tag{3.30}$$

and

$$\begin{aligned}|R_{12}^n|\le h_\Delta\sum_{\vec{j}\in J_h'}\Big|&2\left(|\tilde{u}^{n-1}_{\vec{j}}|^2-|\tilde{U}^{n-1}_{\vec{j}}|^2\right)+2\beta\left(|\tilde{v}^{n-1}_{\vec{j}}|^2-|\tilde{V}^{n-1}_{\vec{j}}|^2\right)\\&+\left(|\tilde{u}^{n-2}_{\vec{j}}|^2-|\tilde{U}^{n-2}_{\vec{j}}|^2\right)+\beta\left(|\tilde{v}^{n-2}_{\vec{j}}|^2-|\tilde{V}^{n-2}_{\vec{j}}|^2\right)\Big|\\&\times\left(|\tilde{u}^n_{\vec{j}}|+|e^n_{\vec{j}}|\right)|e^n_{\vec{j}}|,\end{aligned} \tag{3.31}$$

where the similar term can be estimated as

$$\begin{aligned}\left|\tilde{u}_{\vec{j}}^{n-1}\right|^2-\left|\tilde{U}_{\vec{j}}^{n-1}\right|^2 &\le \left(|\tilde{u}_{\vec{j}}^{n-1}|+|\tilde{U}_{\vec{j}}^{n-1}|\right)\left(|\tilde{u}_{\vec{j}}^{n-1}|-|\tilde{U}_{\vec{j}}^{n-1}|\right)\\ &\le \left(2|\tilde{u}_{\vec{j}}^{n-1}|+|e_{\vec{j}}^{n-1}|\right)|e_{\vec{j}}^{n-1}|\\ &= 2|\tilde{u}_{\vec{j}}^{n-1}||e_{\vec{j}}^{n-1}|+|e_{\vec{j}}^{n-1}|^2.\end{aligned}$$

Applying it into (3.31) yields

$$\begin{aligned}|R_{12}^n| \le h_\Delta \sum_{\vec{j}\in J_h'} &\Big[2\left(2|\tilde{u}_{\vec{j}}^{n-1}||e_{\vec{j}}^{n-1}|+|\tilde{u}_{\vec{j}}^{n-2}||e_{\vec{j}}^{n-2}|\right)\\ &+2\beta\left(2|\tilde{v}_{\vec{j}}^{n-1}||\eta_{\vec{j}}^{n-1}|+|\tilde{v}_{\vec{j}}^{n-2}||\eta_{\vec{j}}^{n-2}|\right)\\ &+\left(2|e_{\vec{j}}^{n-1}|^2+|e_{\vec{j}}^{n-2}|^2\right)\\ &+\beta\left(2|\eta_{\vec{j}}^{n-1}|^2+|\eta_{\vec{j}}^{n-2}|^2\right)\Big]\left(|\tilde{u}_{\vec{j}}^{n}|+|e_{\vec{j}}^{n}|\right)|e_{\vec{j}}^{n}|\\ \le C&\left(\|e^{n-1}\|^2+\|e^{n-2}\|^2+\|\eta^{n-1}\|^2+\|\eta^{n-2}\|^2\right)+C\|e^n\|^2\\ &+C\left(\|e^{n-1}\|_{L^3}^2+\|e^{n-2}\|_{L^3}^2+\|\eta^{n-1}\|_{L^3}^2+\|\eta^{n-2}\|_{L^3}^2\right)\|e^n\|_{L^6}^2,\end{aligned} \tag{3.32}$$

where Hölder inequality has been used in the last step.

The first term on the left-hand side of (3.28) can be estimated as

$$\begin{aligned}\Re\langle D_\tau^\alpha e^n, e^n\rangle &= \Re\left\langle a_{n,0}e^n-\sum_{k=1}^{n-1}\left(a_{n,n-k-1}-a_{n,n-k}\right)e^k-a_{n,n-1}e^0, e^n\right\rangle\\ &= \Re\langle a_{n,0}e^n,e^n\rangle-\sum_{k=1}^{n-1}\Re\langle\left(a_{n,n-k-1}-a_{n,n-k}\right)e^k,e^n\rangle-\Re\langle a_{n,n-1}e^0,e^n\rangle\\ &\ge a_{n,0}\|e^n\|^2-\frac{1}{2}\sum_{k=1}^{n-1}\left(a_{n,n-k-1}-a_{n,n-k}\right)\|e^n\|^2-\frac{1}{2}a_{n,n-1}\|e^n\|^2\\ &\quad -\frac{1}{2}\sum_{k=1}^{n-1}\left(a_{n,n-k-1}-a_{n,n-k}\right)\|e^k\|^2-\frac{1}{2}a_{n,n-1}\|e^0\|^2\\ &= \frac{1}{2}D_\tau^\alpha\|e^n\|^2.\end{aligned} \tag{3.33}$$

With the above inequalities (3.29)-(3.33), (3.28) deduces to

$$\begin{aligned}\frac{1}{2}D_\tau^\alpha\|e^n\|^2 \le\, &C\left(\|e^{n-1}\|^2+\|e^{n-2}\|^2+\|\eta^{n-1}\|^2+\|\eta^{n-2}\|^2\right)+C\|e^n\|^2\\ &+C\left(\|e^{n-1}\|_{L^3}^2+\|e^{n-2}\|_{L^3}^2+\|\eta^{n-1}\|_{L^3}^2+\|\eta^{n-2}\|_{L^3}^2\right)\|e^n\|_{L^6}^2\\ &+\left(\|P^n\|+\|\gamma_1^n\|\right)\|e^n\|.\end{aligned} \tag{3.34}$$

Similarly, multiplying (3.17) by $\eta_{\vec{j}}^n$ and summing them up for $\vec{j} \in J_h'$ gives

$$\mathbf{i}\langle D_\tau^\alpha \eta^n, \eta^n\rangle - \|\nabla_h \eta^n\|^2 + R_2^n = \langle Q^n + \gamma_2^n, \eta^n\rangle, \tag{3.35}$$

where

$$R_2^n = h_\Delta \sum_{\vec{j}\in J_h'} \left[\hat{G}^n(\tilde{v},\tilde{u})\tilde{v}_{\vec{j}}^n - \hat{G}^n(\tilde{V},\tilde{U})\tilde{V}_{\vec{j}}^n\right]\eta_{\vec{j}}^n.$$

Taking the imaginary part of (3.35), we obtain

$$\Re\langle D_\tau^\alpha \eta^n, \eta^n\rangle + \Im(R_2^n) = \Im\langle Q^n + \gamma_2^n, \eta^n\rangle. \tag{3.36}$$

We rewrite R_2^n by

$$\begin{aligned} R_2^n &= h_\Delta \sum_{\vec{j}\in J_h'} \left[\big(\hat{G}^n(\tilde{v},\tilde{u}) - \hat{G}^n(\tilde{V},\tilde{U})\big)\tilde{V}_{\vec{j}}^n + \hat{G}^n(\tilde{v},\tilde{u})\eta_{\vec{j}}^n\right]\eta_{\vec{j}}^n \\ &= h_\Delta \sum_{\vec{j}\in J_h'} \hat{G}^n(\tilde{v},\tilde{u})\big|\eta_{\vec{j}}^n\big|^2 + h_\Delta \sum_{\vec{j}\in J_h'} \big(\hat{G}^n(\tilde{v},\tilde{u}) - \hat{G}^n(\tilde{V},\tilde{U})\big)\big(\tilde{v}_{\vec{j}}^n - \eta_{\vec{j}}^n\big)\eta_{\vec{j}}^n \\ &=: R_{21}^n + R_{22}^n, \end{aligned} \tag{3.37}$$

from which, it is easy to obtain

$$\Im(R_{21}^n) = 0, \tag{3.38}$$

and

$$\begin{aligned} |R_{22}^n| \le{}& h_\Delta \sum_{\vec{j}\in J_h'} \bigg| 2\left(|\tilde{v}_{\vec{j}}^{n-1}|^2 - |\tilde{V}_{\vec{j}}^{n-1}|^2\right) + 2\beta\left(|\tilde{u}_{\vec{j}}^{n-1}|^2 - |\tilde{U}_{\vec{j}}^{n-1}|^2\right) \\ &\qquad + \left(|\tilde{v}_{\vec{j}}^{n-2}|^2 - |\tilde{V}_{\vec{j}}^{n-2}|^2\right) + \beta\left(|\tilde{u}_{\vec{j}}^{n-2}|^2 - |\tilde{U}_{\vec{j}}^{n-2}|^2\right)\bigg| \left(|\tilde{v}_{\vec{j}}^n| + |\eta_{\vec{j}}^n|\right)|\eta_{\vec{j}}^n| \\ \le{}& C\left(\|\eta^{n-1}\|^2 + \|\eta^{n-2}\|^2 + \|e^{n-1}\|^2 + \|e^{n-2}\|^2\right) + C\|\eta^n\|^2 \\ &+ C\left(\|\eta^{n-1}\|_{L^3}^2 + \|\eta^{n-2}\|_{L^3}^2 + \|e^{n-1}\|_{L^3}^2 + \|e^{n-2}\|_{L^3}^2\right)\|\eta^n\|_{L^6}^2. \end{aligned} \tag{3.39}$$

With the inequalities (3.37)-(3.39) and using the same approach above, (3.36) deduces to

$$\begin{aligned} \frac{1}{2}D_\tau^\alpha\|\eta^n\|^2 \le{}& C\left(\|\eta^{n-1}\|^2 + \|\eta^{n-2}\|^2 + \|e^{n-1}\|^2 + \|e^{n-2}\|^2\right) + C\|\eta^n\|^2 \\ &+ C\left(\|\eta^{n-1}\|_{L^3}^2 + \|\eta^{n-2}\|_{L^3}^2 + \|e^{n-1}\|_{L^3}^2 + \|e^{n-2}\|_{L^3}^2\right)\|\eta^n\|_{L^6}^2 \\ &+ \left(\|Q^n\| + \|\gamma_2^n\|\right)\|\eta^n\|. \end{aligned} \tag{3.40}$$

On the other hand, by taking the real part in (3.27) and (3.35), we obtain

$$\|\nabla_h e^n\|^2 = -\Im\langle D_\tau^\alpha e^n, e^n\rangle + \Re(R_1^n) - \Re\langle P^n + \gamma_1^n, e^n\rangle, \tag{3.41}$$

$$\|\nabla_h \eta^n\|^2 = -\Im\langle D_\tau^\alpha \eta^n, \eta^n\rangle + \Re(R_2^n) - \Re\langle Q^n + \gamma_2^n, \eta^n\rangle. \tag{3.42}$$

Denote

$$T_1^n = D_\tau^\alpha e^n - a_{n,0}e^n, \quad T_2^n = D_\tau^\alpha \eta^n - a_{n,0}\eta^n.$$

Then it follows that

$$\begin{aligned}
\|T_1^n\| &= \left\| -\sum_{k=1}^{n-1}\left(a_{n,n-k-1}-a_{n,n-k}\right)e^k - a_{n,n-1}e^0 \right\| \\
&\le \left[\sum_{k=1}^{n-1}\left(a_{n,n-k-1}-a_{n,n-k}\right)+a_{n,n-1}\right]\max_{0\le k'\le n-1}\|e^{k'}\| \\
&\le a_{n,0}\sqrt{K_1}\left(\tau_s^{2-\alpha}+h^2\right) \\
&\le \frac{2^{\alpha+1/\alpha-2}\tau_s^{-1}}{\alpha^{1-\alpha}\Gamma(2-\alpha)}\sqrt{K_1}\left(\tau_s^{2-\alpha}+h^2\right) \\
&\le C\tau_s^{-1}\left(\tau_s^{2-\alpha}+h^2\right),
\end{aligned} \tag{3.43}$$

where Lemma 3.2 has been used in the last step. Similarly,

$$\|T_2^n\| \le C\tau_s^{-1}\left(\tau_s^{2-\alpha}+h^2\right). \tag{3.44}$$

Since

$$\Im\langle D_\tau^\alpha e^n, e^n\rangle = \Im\langle T_1^n, e^n\rangle, \tag{3.45}$$

and by (3.29), we further have

$$|R_{11}^n| \le C\|e^n\|^2.$$

From which together with (3.43), (3.45) and (3.32), (3.41) can be derived to

$$\begin{aligned}
\|\nabla_h e^n\|^2 &= -\Im\langle T_1^n, e^n\rangle + \Re(R_1^n) - \Re\langle P^n+\gamma_1^n, e^n\rangle \\
&\le \left(\|T_1^n\|+\|P^n\|+\|\gamma_1^n\|\right)\|e^n\| + |R_{11}^n| + |R_{12}^n| \\
&\le \left(\|T_1^n\|+\|P^n\|+\|\gamma_1^n\|\right)\|e^n\| + C\|e^n\|^2 \\
&\quad + C\left(\|e^{n-1}\|^2+\|e^{n-2}\|^2+\|\eta^{n-1}\|^2+\|\eta^{n-2}\|^2\right) \\
&\quad + C\left(\|e^{n-1}\|_{L^3}^2+\|e^{n-2}\|_{L^3}^2+\|\eta^{n-1}\|_{L^3}^2+\|\eta^{n-2}\|_{L^3}^2\right)\|e^n\|_{L^6}^2.
\end{aligned} \tag{3.46}$$

Applying the above approach to (3.42), we can similarly get

$$\begin{aligned}
\|\nabla_h \eta^n\|^2 &\le \left(\|T_2^n\|+\|Q^n\|+\|\gamma_2^n\|\right)\|\eta^n\| + C\|\eta^n\|^2 \\
&\quad + C\left(\|\eta^{n-1}\|^2+\|\eta^{n-2}\|^2+\|e^{n-1}\|^2+\|e^{n-2}\|^2\right) \\
&\quad + C\left(\|\eta^{n-1}\|_{L^3}^2+\|\eta^{n-2}\|_{L^3}^2+\|e^{n-1}\|_{L^3}^2+\|e^{n-2}\|_{L^3}^2\right)\|\eta^n\|_{L^6}^2.
\end{aligned} \tag{3.47}$$

Next, we show the error estimates hold in two different cases, i.e., $\tau_s \le h^{2/(2-\alpha)}$ or $h \le \tau_s^{(2-\alpha)/2}$.

Case 1. $\tau_s \le h^{2/(2-\alpha)}$. Based on the induction hypothesis, we use Lemma 3.6 to get

$$
\begin{aligned}
&\|e^m\|_{L^3} \le Ch^{-1/2}\|e^m\|_{L^2} \le Ch^{-1/2}K_1^{1/2}\left(\tau_s^{2-\alpha}+h^2\right) \le 2CK_1^{1/2}h^{3/2}, \quad 1\le m\le n-1,\\
&\|e^n\|_{L^6} \le Ch^{-1}\|e^n\|_{L^2},\\
&\|\eta^m\|_{L^3} \le Ch^{-1/2}\|\eta^m\|_{L^2} \le Ch^{-1/2}K_1^{1/2}\left(\tau_s^{2-\alpha}+h^2\right) \le 2CK_1^{1/2}h^{3/2}, \quad 1\le m\le n-1,\\
&\|\eta^n\|_{L^6} \le Ch^{-1}\|\eta^n\|_{L^2}.
\end{aligned}
$$

When $h \le h_0 = (16C^4K_1)^{-1}$, there are

$$
\left(\|e^{n-1}\|_{L^3}^2+\|e^{n-2}\|_{L^3}^2+\|\eta^{n-1}\|_{L^3}^2+\|\eta^{n-2}\|_{L^3}^2\right)\|e^n\|_{L^6}^2 \le \|e^n\|_{L^2}^2, \tag{3.48}
$$

$$
\left(\|\eta^{n-1}\|_{L^3}^2+\|\eta^{n-2}\|_{L^3}^2+\|e^{n-1}\|_{L^3}^2+\|e^{n-2}\|_{L^3}^2\right)\|\eta^n\|_{L^6}^2 \le \|\eta^n\|_{L^2}^2. \tag{3.49}
$$

Applying (3.48) into (3.34) yields

$$
\begin{aligned}
\frac{1}{2}D_\tau^\alpha\|e^n\|^2 \le{}& C\left(\|e^{n-1}\|^2+\|e^{n-2}\|^2+\|\eta^{n-1}\|^2+\|\eta^{n-2}\|^2\right)\\
&+C\|e^n\|^2+\left(\|P^n\|+\|\gamma_1^n\|\right)\|e^n\|,
\end{aligned} \tag{3.50}
$$

and applying (3.49) into (3.40) similarly yields

$$
\begin{aligned}
\frac{1}{2}D_\tau^\alpha\|\eta^n\|^2 \le{}& C\left(\|\eta^{n-1}\|^2+\|\eta^{n-2}\|^2+\|e^{n-1}\|^2+\|e^{n-2}\|^2\right)\\
&+C\|\eta^n\|^2+\left(\|Q^n\|+\|\gamma_2^n\|\right)\|\eta^n\|.
\end{aligned} \tag{3.51}
$$

Adding the Eqs. (3.50)-(3.51) together, we obtain

$$
\begin{aligned}
&D_\tau^\alpha\left(\|e^n\|^2+\|\eta^n\|^2\right)\\
&\le 4C\left(\|e^{n-1}\|^2+\|e^{n-2}\|^2+\|\eta^{n-1}\|^2+\|\eta^{n-2}\|^2\right)+2C(\|e^n\|^2+\|\eta^n\|^2)\\
&\quad +2\left(\|P^n\|+\|\gamma_1^n\|\right)\|e^n\|+2\left(\|Q^n\|+\|\gamma_2^n\|\right)\|\eta^n\|\\
&\le 4C\left(\|e^{n-1}\|^2+\|e^{n-2}\|^2+\|\eta^{n-1}\|^2+\|\eta^{n-2}\|^2\right)+2C(\|e^n\|^2+\|\eta^n\|^2)\\
&\quad +2\left(\|P^n\|+\|\gamma_1^n\|+\|Q^n\|+\|\gamma_2^n\|\right)\sqrt{\|e^n\|^2+\|\eta^n\|^2}.
\end{aligned}
$$

According to (2.14) and applying Lemma 3.7 with the substitutions

$$
\begin{aligned}
&v^n=\sqrt{\|e^n\|^2+\|\eta^n\|^2},\\
&\lambda_0=2C,\quad \lambda_1=\lambda_2=4C,\quad \lambda_l=0\ \ (l\ge 3),\\
&\mu=2,\quad R=\|P^n\|+\|Q^n\|,
\end{aligned}
$$

we can conclude that

$$
\sqrt{\|e^n\|^2+\|\eta^n\|^2} \le 2E_\alpha(20Cs_n)\left(\sqrt{\|e^0\|^2+\|\eta^0\|^2}+4C_1^*\tau_s^{2-\alpha}+\frac{2\sqrt{C}s_n}{\Gamma(1+\alpha)}\left(\tau_s^2+h^2\right)\right).
$$

Then by (3.20), we get

$$\|e^n\|^2 + \|\eta^n\|^2 \le C\left(\tau_s^{2-\alpha} + h^2\right)^2. \tag{3.52}$$

Applying (3.48) into (3.46) yields

$$\begin{aligned}\|\nabla_h e^n\|^2 \le& \left(\|T_1^n\| + \|P^n\| + \|\gamma_1^n\|\right)\|e^n\| + C\|e^n\|^2 \\ &+ C\left(\|e^{n-1}\|^2 + \|e^{n-2}\|^2 + \|\eta^{n-1}\|^2 + \|\eta^{n-2}\|^2\right),\end{aligned} \tag{3.53}$$

and applying (3.49) into (3.47) yields

$$\begin{aligned}\|\nabla_h \eta^n\|^2 \le& \left(\|T_2^n\| + \|Q^n\| + \|\gamma_2^n\|\right)\|\eta^n\| + C\|\eta^n\|^2 \\ &+ C\left(\|\eta^{n-1}\|^2 + \|\eta^{n-2}\|^2 + \|e^{n-1}\|^2 + \|e^{n-2}\|^2\right).\end{aligned} \tag{3.54}$$

Then together with (2.14), (3.26), (3.43), (3.52) and Lemma 3.5, (3.53) can be derived to

$$\tau_s\|\nabla_h e^n\|^2 \le C\left(\tau_s^{2-\alpha} + h^2\right)^2.$$

Similarly, we have

$$\tau_s\|\nabla_h \eta^n\|^2 \le C\left(\tau_s^{2-\alpha} + h^2\right)^2.$$

The inequality (3.26) is proved in this case.

Case 2. $h \le \tau_s^{(2-\alpha)/2}$. We still assume that (3.26) holds when $1 \le m \le n-1$. According to induction hypothesis and use Lemma 3.6, we can get

$$\begin{aligned}\|e^m\|_{L^6}^2 &\le C\left(\|\nabla_h e^m\| + \|e^m\|\right)^2, \\ \|e^m\|_{L^3}^2 &\le \|e^m\|\|e^m\|_{L^6} \\ &\le C\|e^m\|(\|\nabla_h e^m\| + \|e^m\|) \\ &\le C\tau_s^{2-\alpha}\left(\tau_s^{2-\alpha} + \tau_s^{(3-2\alpha)/2}\right) \\ &\le C\tau_s^{(7-4\alpha)/2}, \quad 1 \le m \le n-1.\end{aligned}$$

When

$$\tau_s \le \tau_s^{(1)} = \min\left\{(16C^2)^{-2/(5-4\alpha)}, 1\right\},$$

we have

$$\begin{aligned}&4C\|e^m\|_{L^3}^2\|e^n\|_{L^6}^2 \\ \le\ & 8C^2\tau_s^{(7-4\alpha)/2}\left(\|\nabla_h e^n\|^2 + \|e^n\|^2\right) \\ \le\ & \frac{1}{2}\tau_s\|\nabla_h e^n\|^2 + \|e^n\|^2, \quad 1 \le m \le n-1.\end{aligned}$$

Then it follows that

$$\begin{aligned}&C\left(\|e^{n-1}\|_{L^3}^2 + \|e^{n-2}\|_{L^3}^2 + \|\eta^{n-1}\|_{L^3}^2 + \|\eta^{n-2}\|_{L^3}^2\right)\|e^n\|_{L^6}^2 \\ \le\ & \frac{1}{2}\tau_s\|\nabla_h e^n\|^2 + \|e^n\|^2,\end{aligned} \tag{3.55}$$

and

$$\begin{aligned}&C\left(\|\eta^{n-1}\|_{L^3}^2+\|\eta^{n-2}\|_{L^3}^2+\|e^{n-1}\|_{L^3}^2+\|e^{n-2}\|_{L^3}^2\right)\|\eta^n\|_{L^6}^2\\&\leq\frac{1}{2}\tau_s\|\nabla_h\eta^n\|^2+\|\eta^n\|^2.\end{aligned}\tag{3.56}$$

Applying (3.55) into (3.34) yields

$$\begin{aligned}\frac{1}{2}D_\tau^\alpha\|e^n\|^2\leq{}&C\left(\|e^{n-1}\|^2+\|e^{n-2}\|^2+\|\eta^{n-1}\|^2+\|\eta^{n-2}\|^2\right)+C\|e^n\|^2\\&+\left(\|P^n\|+\|\gamma_1^n\|\right)\|e^n\|+\frac{1}{2}\tau_s\|\nabla_he^n\|^2,\end{aligned}\tag{3.57}$$

and applying (3.56) into (3.40) similarly yields

$$\begin{aligned}\frac{1}{2}D_\tau^\alpha\|\eta^n\|^2\leq{}&C\left(\|\eta^{n-1}\|^2+\|\eta^{n-2}\|^2+\|e^{n-1}\|^2+\|e^{n-2}\|^2\right)+C\|\eta^n\|^2\\&+\left(\|Q^n\|+\|\gamma_2^n\|\right)\|\eta^n\|+\frac{1}{2}\tau_s\|\nabla_h\eta^n\|^2.\end{aligned}\tag{3.58}$$

It can be further inferred from (3.55) that

$$\begin{aligned}&C\left(\|e^{n-1}\|_{L^3}^2+\|e^{n-2}\|_{L^3}^2+\|\eta^{n-1}\|_{L^3}^2+\|\eta^{n-2}\|_{L^3}^2\right)\|e^n\|_{L^6}^2\leq\frac{1}{2}\|\nabla_he^n\|^2+\|e^n\|^2,\\&C\left(\|\eta^{n-1}\|_{L^3}^2+\|\eta^{n-2}\|_{L^3}^2+\|e^{n-1}\|_{L^3}^2+\|e^{n-2}\|_{L^3}^2\right)\|\eta^n\|_{L^6}^2\leq\frac{1}{2}\|\nabla_h\eta^n\|^2+\|\eta^n\|^2.\end{aligned}$$

Then the Eqs. (3.46) and (3.47) can be estimated as

$$\begin{aligned}\tau_s\|\nabla_he^n\|^2\leq{}&\tau_s\|T_1^n\|\|e^n\|+\tau_s\left(\|P^n\|+\|\gamma_1^n\|\right)\|e^n\|\\&+\tau_s\Big(C\|e^n\|^2+C\left(\|e^{n-1}\|^2+\|e^{n-2}\|^2+\|\eta^{n-1}\|^2+\|\eta^{n-2}\|^2\right)\Big)\\&+\tau_s\left(\frac{1}{2}\|\nabla_he^n\|^2+\|e^n\|^2\right),\\\tau_s\|\nabla_h\eta^n\|^2\leq{}&\tau_s\|T_2^n\|\|\eta^n\|+\tau_s\left(\|Q^n\|+\|\gamma_2^n\|\right)\|\eta^n\|\\&+\tau_s\Big(C\|\eta^n\|^2+C\left(\|\eta^{n-1}\|^2+\|\eta^{n-2}\|^2+\|e^{n-1}\|^2+\|e^{n-2}\|^2\right)\Big)\\&+\tau_s\left(\frac{1}{2}\|\nabla_h\eta^n\|^2+\|\eta^n\|^2\right).\end{aligned}$$

Combining with (3.43) and (3.44), it implies that

$$\begin{aligned}\frac{1}{2}\tau_s\|\nabla_he^n\|^2\leq{}&\tau_s^{2-\alpha}\|e^n\|+\tau_s\left(\|P^n\|+\|\gamma_1^n\|\right)\|e^n\|\\&+\tau_s\Big(C\|e^n\|^2+C\left(\|e^{n-1}\|^2+\|e^{n-2}\|^2+\|\eta^{n-1}\|^2+\|\eta^{n-2}\|^2\right)\Big),\end{aligned}\tag{3.59}$$

$$\begin{aligned}\frac{1}{2}\tau_s\|\nabla_h\eta^n\|^2\leq{}&\tau_s^{2-\alpha}\|\eta^n\|+\tau_s\left(\|Q^n\|+\|\gamma_2^n\|\right)\|\eta^n\|\\&+\tau_s\Big(C\|\eta^n\|^2+C\left(\|\eta^{n-1}\|^2+\|\eta^{n-2}\|^2+\|e^{n-1}\|^2+\|e^{n-2}\|^2\right)\Big).\end{aligned}\tag{3.60}$$

Summing (3.59)-(3.60) and (3.57)-(3.58) together, respectively. Then adding the two equations together, we obtain

$$
\begin{aligned}
& D_\tau^\alpha \left(\|e^n\|^2 + \|\eta^n\|^2\right) \\
\le\ & 4C(1+\tau_s)\left(\|e^{n-1}\|^2 + \|e^{n-2}\|^2 + \|\eta^{n-1}\|^2 + \|\eta^{n-2}\|^2\right) \\
& + 2C(1+\tau_s)(\|e^n\|^2 + \|\eta^n\|^2) \\
& + 2\left((1+\tau_s)(\|P^n\| + \|\gamma_1^n\|) + \tau_s^{2-\alpha}\right)\|e^n\| \\
& + 2\left((1+\tau_s)(\|Q^n\| + \|\gamma_2^n\|) + \tau_s^{2-\alpha}\right)\|\eta^n\| \\
\le\ & 8C\left(\|e^{n-1}\|^2 + \|e^{n-2}\|^2 + \|\eta^{n-1}\|^2 + \|\eta^{n-2}\|^2\right) + 4C(\|e^n\|^2 + \|\eta^n\|^2) \\
& + 4\left(\|P^n\| + \|\gamma_1^n\| + \|Q^n\| + \|\gamma_2^n\| + \tau_s^{2-\alpha}\right)\sqrt{\|e^n\|^2 + \|\eta^n\|^2}.
\end{aligned}
$$

By using Lemma 3.7 and (2.14) again, we can similarly conclude that

$$
\|e^n\|^2 + \|\eta^n\|^2 \le C\left(\tau_s^{2-\alpha} + h^2\right)^2, \tag{3.61}
$$

when under the circumstance $\tau_s \le \min\{\tau_s^{(2)}, 1\}$.

Then combining (2.14), (3.43), (3.52), (3.26) and Lemma 3.5, (3.59) can be derived to

$$
\tau_s\|\nabla_h e^n\|^2 \le C\left(\tau_s^{2-\alpha} + h^2\right)^2,
$$

and (3.60) can be similarly deduced as

$$
\tau_s\|\nabla_h \eta^n\|^2 \le C\left(\tau_s^{2-\alpha} + h^2\right)^2.
$$

The result of inequality (3.26) is proved in this case.

Setting $\tau_s^* = \min\{\tau_s^{(1)}, \tau_s^{(2)}\}$. Up to now, we have proved that (3.26) holds for $m = n$ when $h \le h_0$ and $\tau_s \le \tau_s^*$. We complete the induction.

For the other condition, when

$$
\tau_s^{2-\alpha} + h^2 \ge C_{h,\tau}
$$

for some positive constant $C_{h,\tau}$, by applying Lemma 3.4, we can get

$$
\begin{aligned}
& \|\tilde{u}^n - \tilde{U}^n\| + \|\tilde{v}^n - \tilde{V}^n\| \\
\le\ & \|\tilde{u}^0\| + \|\tilde{U}^0\| + \|\tilde{v}^0\| + \|\tilde{V}^0\| \\
\le\ & \frac{\|\tilde{u}^0\| + \|\tilde{U}^0\| + \|\tilde{v}^0\| + \|\tilde{V}^0\|}{C_{h,\tau}}\left(\tau_s^{2-\alpha} + h^2\right).
\end{aligned}
$$

Setting

$$
C^* = \max\left\{K_1, (\|\tilde{u}^0\| + \|\tilde{U}^0\| + \|\tilde{v}^0\| + \|\tilde{V}^0\|)/C_{h,\tau}\right\},
$$

finishes the proof of Theorem 2.1. ☐

4. An alternative approach to error analysis

In this section, we get the unconditional convergence results by using the so-called temporal-spatial splitting method. We present a routing way to obtain the unconditional convergence results. To show the idea conveniently, here the spatial discretization is done by using the finite element methods. Let P_h be a quasiuniform partition of Ω into triangles $\mathcal{T}_k$ $(1 \leq k \leq M)$ in R^2 or tetrahedra in R^3. Denote $h_{FE} = \max_{1\leq k\leq M}\{\mathrm{diam}\mathcal{T}_k\}$ as the spatial mesh size. Let S_h be the finite-dimensional subspace of $H_0^1(\Omega)$, which consists of continuous piecewise polynomials of degree r $(r \geq 1)$ on P_h.

The temporal discretization still employs the transformed L1 scheme. We divided the interval $[0,T^\alpha]$ into N equal subintervals with time step $\tau_s = T^\alpha/N$ and let $s_n = n\tau_s$. Denote $\tilde{u}^n = \tilde{u}(\vec{x}, s_n)$ and $\tilde{v}^n = \tilde{v}(\vec{x}, s_n)$, $0 \leq n \leq N$, which are the exact solutions after the change of variable $t = s^{1/\alpha}$. With above definitions and notations, the transformed L1-Galerkin FEM is to seek $(\tilde{U}_h^n, \tilde{V}_h^n) \in S_h \times S_h$ such that

$$\mathbf{i}(D_\tau^\alpha \tilde{U}_h^n, \omega_{1h}) - (\nabla_h \tilde{U}_h^n, \nabla\omega_{1h}) + \big(f_1(\hat{U}_h^n, \hat{V}_h^n)\tilde{U}_h^n, \omega_{1h}\big) = 0, \quad 1 \leq n \leq N, \tag{4.1}$$

$$\mathbf{i}(D_\tau^\alpha \tilde{V}_h^n, \omega_{2h}) - (\nabla_h \tilde{V}_h^n, \nabla\omega_{2h}) + \big(f_2(\hat{U}_h^n, \hat{V}_h^n)\tilde{V}_h^n, \omega_{2h}\big) = 0, \quad 1 \leq n \leq N \tag{4.2}$$

for any $(\omega_{1h}, \omega_{2h}) \in S_h \times S_h$, where $\hat{U}_h^1 = R_h\tilde{u}_0$, $\hat{V}_h^1 = R_h\tilde{v}_0$ with R_h being the usual Ritz projection operator, and

$$\hat{U}_h^n = 2\tilde{U}_h^{n-1} - \tilde{U}_h^{n-2}, \quad \hat{V}_h^n = 2\tilde{V}_h^{n-1} - \tilde{V}_h^{n-2}, \quad n \geq 2.$$

We can obtain the unconditional convergence results for the proposed transformed L1-Galerkin FEM scheme (4.1)-(4.2).

Theorem 4.1. *Suppose that system* (2.1)-(2.2) *admits a unique solution* $(\tilde{u}, \tilde{v}) \in [H^{r+1}(\Omega) \cap H_0^1(\Omega)]^2$ *satisfying* (2.3)-(2.4). *Then the proposed r-degree finite element scheme* (4.1)-(4.2) *admits a unique solution* $(\tilde{U}_h^n, \tilde{V}_h^n), n = 1, 2, \ldots, N$, *satisfying*

$$\|\tilde{u}^n - \tilde{U}_h^n\|_{L^2}^2 + \|\tilde{v}^n - \tilde{V}_h^n\|_{L^2}^2 \leq C_f^* \left(\tau_s^{2-\alpha} + h_{FE}^{r+1}\right)^2, \quad 1 \leq n \leq N, \quad r = 1, 2,$$

where C_f^* *is a positive constant independent of* τ_s *and* h_{FE}.

Now, we follow the proof the unconditional convergent results in [25] and show the main idea to get the results. First, let $(\tilde{U}^n, \tilde{V}^n)$, $n = 1, 2, \ldots, N$ be the numerical solutions for the time semi-discrete system

$$\mathbf{i}D_\tau^\alpha \tilde{U}^n + \Delta\tilde{U}^n + f_1(\hat{U}^n, \hat{V}^n)\tilde{U}^n = 0, \quad 1 \leq n \leq N, \tag{4.3}$$

$$\mathbf{i}D_\tau^\alpha \tilde{V}^n + \Delta\tilde{V}^n + f_2(\hat{U}^n, \hat{V}^n)\tilde{V}^n = 0, \quad 1 \leq n \leq N, \tag{4.4}$$

where $\hat{U}^1 = \tilde{u}^0$, $\hat{V}^1 = \tilde{v}^0$, and

$$\hat{U}^n = 2\tilde{U}^{n-1} - \tilde{U}^{n-2}, \quad \hat{V}^n = 2\tilde{V}^{n-1} - \tilde{V}^{n-2}, \quad n \geq 2.$$

One can show that the time-discrete system defined in (4.3)-(4.4) has a unique solution $(\tilde{U}^n, \tilde{V}^n)$, $n = 1, 2, \dots, N$. And there exists $\tau^* > 0$ such that when $\tau_s \le \tau^*$,

$$\|\tilde{u}^n - \tilde{U}^n\|_{H^2} + \|\tilde{v}^n - \tilde{V}^n\|_{H^2} \le \tau_s^{1-\alpha}, \quad 1 \le n \le N, \tag{4.5}$$

$$\|\tilde{U}^n\|_{L^\infty} + \|\tilde{V}^n\|_{L^\infty} \le 2K_2, \qquad 1 \le n \le N, \tag{4.6}$$

where K_2 is a positive constant independent of τ_s and n. Thanks to boundedness of $\|\tilde{U}^n\|_{L^\infty}$ and $\|\tilde{V}^n\|_{L^\infty}$, we can obtain that $\max_{1\le n\le N} \|R_h\tilde{U}^n\|_{L^\infty} + \max_{1\le n\le N} \|R_h\tilde{V}^n\|_{L^\infty}$ is bounded. Then we can define

$$K_3 = \max_{1\le n\le N} \|R_h\tilde{U}^n\|_{L^\infty} + \max_{1\le n\le N} \|R_h\tilde{V}^n\|_{L^\infty} + 1.$$

We present a primary error estimates of $R_h\tilde{U}^n - \tilde{U}_h^n$ and $R_h\tilde{V}^n - \tilde{V}_h^n$, then prove the boundedness of numerical solution $(\tilde{U}_h^n, \tilde{V}_h^n)$ in L^∞-norm unconditionally, i.e., the proposed r-degree finite element scheme (4.1)-(4.2) admits a unique solution $(\tilde{U}_h^n, \tilde{V}_h^n)$, $n = 1, 2, \dots, N$. Moreover, there exists $\tau_1^* > 0$ and $h^* > 0$ such that when $\tau_s \le \tau_1^*$, $h_{FE} < h^*$,

$$\|R_h\tilde{U}^n - \tilde{U}_h^n\|_{L^2} + \|R_h\tilde{V}^n - \tilde{V}_h^n\|_{L^2} \le h_{FE}^{\kappa}, \quad 1 \le n \le N, \tag{4.7}$$

$$\|\tilde{U}_h^n\|_{L^\infty} + \|\tilde{V}_h^n\|_{L^\infty} \le 2K_3, \qquad 1 \le n \le N, \tag{4.8}$$

where κ is a constant satisfies $3/2 < \kappa < 2$.

According to the boundedness of the finite element numerical solution $(\tilde{U}_h^n, \tilde{V}_h^n)$, we then can provide error estimates for the fully-discrete system unconditionally. We can eventually get a complete proof of the Theorem 4.1.

5. Numerical experiments

In this section, we carry out several numerical examples to confirm our theoretical results in the paper. All computations are performed with Matlab. Throughout the experiments, the spatial domain is divided into M parts in every direction uniformly, that is, in the multi-dimensional cases, we set $h_i = L_i/M =: h, i = 1, \dots, d$. The time interval $[0, T^\alpha]$ is also divided uniformly into N parts, $\tau_s = T^\alpha/N$. Then we use the discrete L^2-norm to measure the global error of the scheme, namely,

$$E_u(M,N) = \|\tilde{U}^N - u(T)\|_{L^2}, \quad E_v(M,N) = \|\tilde{V}^N - v(T)\|_{L^2}.$$

Example 5.1. We consider the following TF-CNLS system:

$$\mathbf{i}\partial_t^\alpha u + \Delta u + \left(|u|^2 + \beta|v|^2\right) u = f_1, \quad x \in \Omega, \quad 0 < t \le T, \tag{5.1}$$

$$\mathbf{i}\partial_t^\alpha v + \Delta v + \left(|v|^2 + \beta|u|^2\right) v = f_2, \quad x \in \Omega, \quad 0 < t \le T, \tag{5.2}$$

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad x \in \Omega, \tag{5.3}$$

$$u|_{x\in\partial\Omega} = 0, \quad v|_{x\in\partial\Omega} = 0, \tag{5.4}$$

where $\beta = 1$, $\Omega = [0,1] \times [0,1]$, (f_1, f_2) and (u_0, v_0) are chosen correspondingly to the exact solutions given by

$$u(x,t) = 3(t^\alpha + t^2)(1 - x_1)(1 - x_2)\sin x_1 \sin x_2,$$
$$v(x,t) = t^\alpha \sin(\pi x_1)\sin(\pi x_2).$$

The global errors in the terminal time $T = 1$ are listed in Table 1. To evaluate the accuracy in the spatial direction, we fix $N = 1000$. The numerical results with various α are provided in Table 1, which indicates the scheme can have a second-order convergence rate in space. To test the accuracy in the temporal direction, we take $h = \tau_s^{(2-\alpha)/2}$ such that the spatial error bound in L^2-norm is proportional to the temporal error of $\tau_s^{2-\alpha}$. The convergence orders with different α are shown in Table 2. The results are found to be the desired temporal order of convergence $2 - \alpha$, which confirms our theoretical analysis in previous section. We also test our scheme with $h = 1/40$ and large time steps $\tau_s = h, 5h, 10h$ and present error results in Table 3. Numerical results show that the proposed scheme is stable for the large time steps.

In order to confirm the unconditional convergence of the proposed method, we solve the system with different stepsizes. The numerical results can be seen in Fig. 1. These results imply that for each fixed τ_s, the error tends to be a constant. It implies that the errors hold without any time-step restrictions dependent on the spatial mesh size.

Table 1: Spatial convergence rates with $N = 1000$ for Example 5.1.

M	$\alpha = 0.3$		$\alpha = 0.5$		$\alpha = 0.7$	
	$E_u(M,N)$	$order(u)$	$E_u(M,N)$	$order(u)$	$E_u(M,N)$	$order(u)$
10	3.97e-04	*	3.97e-04	*	3.96e-04	*
20	9.87e-05	2.01	9.87e-05	2.01	9.86e-05	2.01
30	4.38e-05	2.00	4.38e-05	2.00	4.38e-05	2.00
40	2.46e-05	2.00	2.46e-05	2.00	2.46e-05	2.00
50	1.57e-05	2.01	1.57e-05	2.00	1.57e-05	2.00
M	$E_v(M,N)$	$order(v)$	$E_v(M,N)$	$order(v)$	$E_v(M,N)$	$order(v)$
10	4.55e-03	*	4.55e-03	*	4.55e-03	*
20	1.13e-03	2.01	1.13e-03	2.01	1.13e-03	2.01
30	5.03e-04	2.00	5.03e-04	2.00	5.03e-04	2.00
40	2.82e-04	2.00	2.83e-04	2.00	2.83e-04	2.00
50	1.81e-04	2.00	1.81e-04	2.00	1.81e-04	2.00

Example 5.2. Consider the TF-CNLS equations (5.1)-(5.4) in three-dimensional space, where $\beta = 1.5$, $\Omega = [0,1] \times [0,1] \times [0,1]$, (f_1, f_2) and (u_0, v_0) are chosen correspondingly to the exact solutions given by

$$u(x,t) = 60(t^\alpha + t^2)(1 - x_1)(1 - x_2)(1 - x_3)\sin x_1 \sin x_2 \sin x_3,$$
$$v(x,t) = t^\alpha \sin(\pi x_1)\sin(\pi x_2)\sin(\pi x_3).$$

Table 2: Temporal convergence rates with $M=[N^{(2-\alpha)/2}]$ for Example 5.1.

N	$\alpha=0.4$		$\alpha=0.6$		$\alpha=0.8$	
	$E_u(M,N)$	$order(u)$	$E_u(M,N)$	$order(u)$	$E_u(M,N)$	$order(u)$
20	2.62e-04	*	5.73e-04	*	1.07e-03	*
40	9.25e-05	1.50	2.23e-04	1.36	4.79e-04	1.16
80	3.22e-05	1.52	8.70e-05	1.36	1.99e-04	1.27
160	1.08e-05	1.57	3.17e-05	1.46	8.89e-05	1.16
N	$E_v(M,N)$	$order(v)$	$E_v(M,N)$	$order(v)$	$E_v(M,N)$	$order(v)$
20	3.53e-03	*	6.98e-03	*	1.27e-02	*
40	1.19e-03	1.57	2.65e-03	1.40	5.60e-03	1.18
80	4.00e-04	1.58	1.02e-03	1.38	2.31e-03	1.28
160	1.30e-04	1.62	3.67e-04	1.47	1.03e-03	1.17

Table 3: Numerical results of Example 5.1 with $\beta=1$ and $h=1/40$.

	$\alpha=0.3$		$\alpha=0.5$		$\alpha=0.7$	
	$E_u(M,N)$	$E_v(M,N)$	$E_u(M,N)$	$E_v(M,N)$	$E_u(M,N)$	$E_v(M,N)$
$\tau_s=h$	2.7941e-05	1.9596e-04	3.4984e-05	2.3575e-04	5.3212e-05	2.4826e-04
$\tau_s=5h$	5.0139e-04	9.3403e-04	4.4858e-04	6.3716e-04	4.8176e-04	4.9059e-04
$\tau_s=10h$	1.3047e-03	2.9126e-03	1.2905e-03	2.6391e-03	1.3380e-03	2.4283e-03

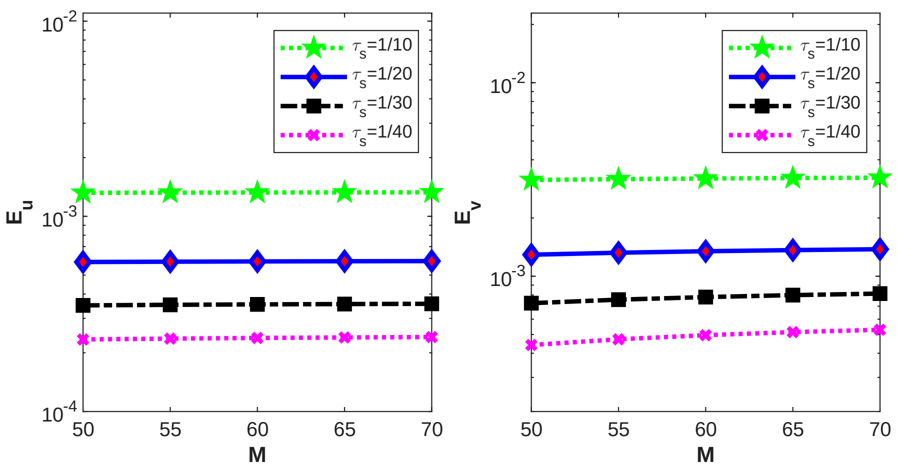

Figure 1: L^2-error of E_u and E_v for Example 5.1 with $\beta=1$, $\alpha=0.2$.

To verify our theoretical results, we solve the three-dimensional problem by taking different values in the spatial discretization with fixed τ_s. The parameters are set as $\beta=1.5$ and $\alpha=0.6$. As can be seen in the Fig. 2, the L^2-errors of E_u and E_v both still converge to a constant for a fixed τ_s, which shows that the scheme is unconditionally convergent without any time-step restrictions.

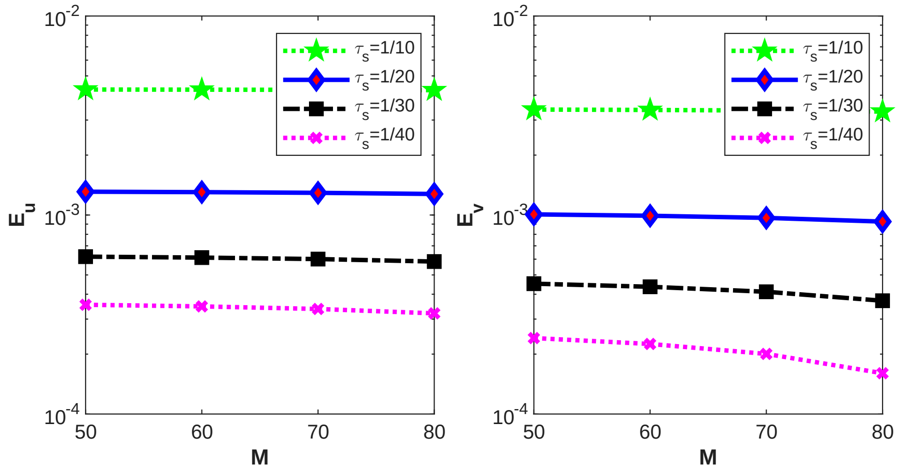

Figure 2: L^2-error of E_u and E_v for Example 5.2 with $\beta = 1.5$, $\alpha = 0.6$.

6. Conclusions

In this paper, we mainly proposed a transformed L1 semi-implicit finite difference method for the time-fractional coupled nonlinear Schrödinger system. The scheme combines the change of variable and the finite difference method. The unique solvability and the unconditionally optimal error estimate of the scheme are further discussed. This method takes the initial singularity into account. We also present a routing way which is the so-called temporal-spatial splitting method to obtain the unconditional convergence results. Both the methods show that the scheme can have $(2-\alpha)$-th order convergence in temporal direction. Several numerical examples are performed to support the theoretical results in the paper.

Acknowledgments

A. S. Hendy gratefully acknowledges the hospitality at the Huazhong University of Science and Technology in the winter of 2024, where part of this work was conducted.

This work is supported in part by the National Natural Science Foundation of China (NSFC) (Grant Nos. 11771162, 12231003).

References

[1] A. A. ALIKHANOV, *A new difference scheme for the time fractional diffusion equation*, J. Comput. Phys. (2015), 424–438.

[2] G. BAI, B. LI, AND Y. WU, *A constructive low-regularity integrator for the 1D cubic nonlinear Schrödinger equation under the Neumann boundary condition*, IMA J. Numer. Anal. 43 (2023), 3243–3281.

[3] W. Bao and Y. Cai, *Uniform error estimates of finite difference methods for the nonlinear Schrödinger equation with wave operator*, SIAM J. Numer. Anal. 50 (2012), 492–521.

[4] A. H. Bhrawy, E. H. Doha, and S. S. Ezz-Eldien, *A new Jacobi spectral collocation method for solving $1+1$ fractional Schrödinger equations and fractional coupled Schrödinger systems*, Eur. Phys. J. Plus 129 (2014), 1–21.

[5] F. E. Browder, *Existence and uniqueness theorems for solutions of nonlinear boundary value problems*, in: Proceedings of Symposia in Applied Mathematics, AMS 17 (1965), 24–49.

[6] H. Brunner, *The numerical solution of weakly singular Volterra integral equations by collocation on graded meshes*, Math. Comput. 45 (1985), 417–437.

[7] H. Brunner, L. Ling, and M. Yamamoto, *Numerical simulations of 2D fractional subdiffusion problems*, J. Comput. Phys. 229 (2010), 6613–6622.

[8] W. Cao et al., *Implicit-explicit difference schemes for nonlinear fractional differential equations with nonsmooth solutions*, SIAM J. Comput. 38 (2016), A3070–A3093.

[9] O. Chepizhko and F. Peruani, *Diffusion, subdiffusion, and trapping of active particles in heterogeneous media*, Phys. Rev. Lett. 111 (2013), 160604.

[10] K. W. Chow, K. K. Y. Wong, and K. Lam, *Modulation instabilities in a system of four coupled, nonlinear Schrödinger equations*, Phys. Lett. A 372 (2008), 4596–4600.

[11] N. J. Ford, M. M. Rodrigues, and N. Vieira, *A numerical method for the fractional Schrödinger type equation of spatial dimension two*, Fract. Calc. Appl. Anal. 16 (2013), 1–15.

[12] X. Guo and M. Xu, *Some physical applications of fractional Schrödinger equation*, J. Math. Phys. 47 (2006), 082104.

[13] A. S. Hendy and M. A. Zaky, *Global consistency analysis of L1-Galerkin spectral schemes for coupled nonlinear space-time fractional Schrödinger equations*, Appl. Numer. Math. 156 (2020), 276–302.

[14] D. Hou, M. T. Hasan, and C. Xu, *Müntz spectral methods for the time-fractional diffusion equation*, J. Comput. Methods Appl. Math. 18 (2018), 43–62.

[15] D. Hou and J. Xu, *A second order energy dissipative scheme for time fractional L2 gradient flows using SAV approach*, J. Sci. Comput. 90 (2022), Article 25.

[16] D. Hou, H. Zhu, and J. Xu, *Highly efficient schemes for time-fractional Allen-Cahn equation using extended SAV approach*, Numer. Algorithms 88 (2021), 1077–1108.

[17] B. Jin, R. Lazarov, and Z. Zhou, *An analysis of the L1 scheme for the subdiffusion equation with nonsmooth data*, IMA J. Numer. Anal. 36 (2016), 197–221.

[18] B. Jin, B. Li, and Z. Zhou, *Correction of high-order BDF convolution quadrature for fractional evolution equations*, SIAM J. Comput. 39 (2017), A3129–A3152.

[19] B. Jin, B. Li, and Z. Zhou, *Numerical analysis of nonlinear subdiffusion equations*, SIAM J. Numer. Anal. 56 (2018), 1–23.

[20] Y. S. Kivshar and G. P. Agrawal, *Optical Solitons: From Fibers to Photonic Crystals*, Academic Press, 2003.

[21] B. Li and W. Sun, *Error analysis of linearized semi-implicit Galerkin finite element methods for nonlinear parabolic equations*, Int. J. Numer. Anal. Model. 10 (2013), 622–633.

[22] B. Li and W. Sun, *Unconditional convergence and optimal error estimates of a Galerkin-mixed FEM for incompressible miscible flow in porous media*, SIAM J. Numer. Anal. 51 (2013), 1959–1977.

[23] C. Li and W. Zhen, *L1/local discontinuous Galerkin method for the time-fractional Stokes equation*, Numer. Math. Theory Methods Appl. 15 (2022), 1099–1127.

[24] D. Li, W. Sun, and C. Wu, *A novel numerical approach to time-fractional parabolic equa-*

tions with nonsmooth solutions, Math. Theor. Meth. Appl. 14 (2021), 355–376.
[25] D. LI, J. WANG, AND J. ZHANG, *Unconditionally convergent L1-Galerkin FEMs for nonlinear time fractional Schrödinger equations*, SIAM J. Sci. Comput. 39 (2017), A3067–A3088.
[26] D. LI, C. WU, AND Z. ZHANG, *Linearized Galerkin FEMs for nonlinear time fractional parabolic problems with non-smooth solutions in time direction*, J. Sci. Comput. 80 (2019), 403–419.
[27] D. LI, J. ZHANG, AND Z. ZHANG, *Unconditionally optimal error estimates of a linearized Galerkin method for nonlinear time fractional reaction-subdiffusion equations*, J. Sci. Comput. 76 (2018), 848–866.
[28] M. LI, X. GU, C. HUANG, M. FEI, AND G. ZHANG, *A fast linearized conservative finite element method for the strongly coupled nonlinear fractional Schrödinger equations*, J. Comput. Phys. 358 (2018), 256–282.
[29] X. LI AND C. XU, *A space-time spectral method for the time fractional diffusion equation*, SIAM J. Numer. Anal. 47 (2009), 2108–2131.
[30] H. LIAO, D. LI, AND J. ZHANG, *Sharp error estimate of the nonuniform L1 formula for linear reaction-subdiffusion equations*, SIAM J. Numer. Anal. 56 (2018), 1112–1133.
[31] H. LIAO, Y. YAN, AND J. ZHANG, *Unconditional convergence of a fast two-level linearized algorithm for semilinear subdiffusion equations*, J. Sci. Comput. 80 (2019), 1–25.
[32] Y. LIN AND C. XU, *Finite difference/spectral approximations for the time-fractional diffusionequation*, J. Comput. Phys. 225 (2007), 1533–1552.
[33] S. MA ET AL., *Mass- and energy-conserving Gauss collocation methods for the nonlinear Schrödinger equation with a wave operator*, Adv. Comput. Math. 77 (2023).
[34] S. V. MANAKOV, *On the theory of two-dimensional stationary self-focusing of electromagnetic waves*, Sov. Phys. JETP 38 (1974), 248–253.
[35] W. MCLEAN, *Regularity of solutions to a time-fractional diffusion equation*, ANZIAM J. 52 (2010), 123–138.
[36] Y. MEI, M. CUI, AND F. ZENG, *A time two-grid algorithm for two-dimensional nonlinear time-fractional partial integro-differential equations*, Math. Comput. Simulation 221 (2024), 550–569.
[37] K. MUSTAPHA, *An implicit finite-difference time-stepping method for a sub-diffusion equation, with spatial discretization by finite elements*, IMA J. Numer. Anal. 31 (2011), 719–739.
[38] N. I. OKPOSO, P. VEERESHA, AND E. N. OKPOSO, *Solutions for time-fractional coupled nonlinear Schrödinger equations arising in optical solitons*, Chinese J. Phys. 77 (2022), 965–984.
[39] H. QIN, D. LI, AND Z. ZHANG, *A novel scheme to capture the initial dramatic evolutions of nonlinear subdiffusion equations*, J. Sci. Comput. 89 (2021), 1–20.
[40] H. QIN, F. WU, AND B. ZHOU, *Alikhanov linearized Galerkin finite element methods for nonlinear time-fractional Schrödinger equations*, J. Comput. Math. 41 (2023), 1305–1324.
[41] J. REN, H. LIAO, J. ZHANG, AND Z. ZHANG, *Sharp H1-norm error estimates of two time-stepping schemes for reaction-subdiffusion problems*, J. Comput. Appl. Math. 389 (2021), 113352.
[42] K. SAKAMOTO AND M. YAMAMOTO, *Initial value/boundary value problems for fractional diffusion-wave equations and applications to some inverse problems*, J. Math. Anal. Appl. 382 (2011), 426–447.
[43] M. STYNES, *A survey of the L1 scheme in the discretisation of time-fractional problems*, Numer. Math. Theor. Meth. Appl. 15 (2022), 1173–1192.
[44] M. STYNES, O. EUGENE, AND J. L. GRACIA, *Error analysis of a finite difference method on*

graded meshes for a time-fractional diffusion equation, SIAM J. Numer. Anal. 55 (2017), 1057–1079.

[45] H. SUN ET AL., *A new collection of real world applications of fractional calculus in science and engineering*, Commun. Nonlinear Sci. Numer. Simul. 64 (2018), 213–231.

[46] W. SUN AND J. WANG, *Optimal error analysis of Crank-Nicolson schemes for a coupled nonlinear Schrödinger system in 3D*, J. Comput. Appl. Math. 317 (2017), 685–699.

[47] J. WANG, *Unconditional stability and convergence of Crank-Nicolson Galerkin FEMs for a nonlinear Schrödinger-Helmholtz system*, Numer. Math. 139 (2018), 479–503.

[48] J. WANG, J. WANG, AND L. YIN, *A single-step correction scheme of Crank-Nicolson convolution quadrature for the subdiffusion equation*, J. Sci. Comput. 87 (2021), Article 26.

[49] Y. XING AND Y. YAN, *A higher order numerical method for time fractional partial differential equations with nonsmooth data*, J. Comput. Phys. 357 (2018), 305–323.

[50] Y. YANG ET AL., *Compatible L^2 norm convergence of variable-step L1 scheme for the time-fractional MBE mobel with slope selection*, J. Comput. Phys. 467 (2022), 111467.

[51] W. YU ET AL., *Control of dark and anti-dark solitons in the (2+1)-dimensional coupled nonlinear Schrödinger equations with perturbed dispersion and nonlinearity in a nonlinear optical system*, Nonlinear Dyn. 97 (2019), 471–483.

[52] W. YUAN, D. LI, AND C. ZHANG, *Linearized transformed L1 Galerkin FEMs with unconditional convergence for nonlinear time fractional Schrödinger equations*, Numer. Math. Theory Methods Appl. 16 (2023), 348–369.

[53] Y. ZHANG, Z. SUN, AND H. LIAO, *Finite difference methods for the time fractional diffusion equation on non-uniform meshes*, J. Comput. Phys. 265 (2014), 195–210.

[54] X. ZHAO, Z. SUN, AND Z. HAO, *A fourth-order compact ADI scheme for two-dimensional nonlinear space fractional Schödinger equation*, SIAM J. Sci. Comput. 36 (2014), A2865–A2886.

[55] X. ZHENG, J. JIA, AND X. GUO, *Eliminating solution singularity of variably distributed-order time-fractional diffusion equation via strongly singular initial distribution*, Chaos Soliton. Fract. 174 (2023), 113908.

[56] B. ZHOU, X. CHEN, AND D. LI, *Nonuniform Alikhanov linearized Galerkin finite element methods for nonlinear time-fractional parabolic equations*, J. Sci. Comput. 85 (2020), Article 39.

[57] Y. T. ZHOU AND M. STYNES, *Optimal convergence rates in time-fractional discretisations: The L1, $\bar{L}1$ and Alikhanov schemes*, East Asian J. Appl. Math. 12 (2022), 503–520.