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Stability and Error Analysis of SAV Semi-Discrete Scheme for Cahn-Hilliard-Navier-Stokes Model

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Abstract. We construct first- and second-order time semi-discretization numerical schemes for the Cahn-Hilliard-Navier-Stokes model. This discretization scheme is based on the energy form of the scalar auxiliary variable approach for the coupling terms of model and pressure correction in the Navier-Stokes equations, which are fully decoupled. Then, we apply the fully explicit forms and the two scalar auxiliary variables to obtain stable unconditional energy over time. At the same time, we present the error analysis for the first-order scheme and the convergence rate for all relevant variables in different norms. Finally, numerical examples are presented to validate the theoretical analysis.

AMS subject classifications: 35Q30, 65M12, 65M60, 65P40

Key words: Cahn-Hilliard-Navier-Stokes, fully decoupled, scalar auxiliary variable (SAV), energy stability, error estimates.

1. Introduction

In this article, we consider the Cahn-Hilliard-Navier-Stokes (CHNS) system as follows:

$$
\begin{aligned}
&\frac{\partial \phi}{\partial t} + (\mathbf{u}\cdot\nabla)\phi - M\Delta\mu = 0 && \text{in } \Omega\times(0,T],\\
&\mu + \lambda\Delta\phi - \lambda G'(\phi) = 0 && \text{in } \Omega\times(0,T],\\
&\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p - \mu\nabla\phi = 0 && \text{in } \Omega\times(0,T],\\
&\nabla\cdot\mathbf{u} = 0 && \text{in } \Omega\times(0,T],
\end{aligned}
\tag{1.1}
$$

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where

$$G(\phi) = \frac{1}{4\epsilon^2}(\phi^2 - 1)^2$$

is a nonlinear free energy density, where ϵ denotes the interface width and $M, \lambda, \nu > 0$, describes the mobility, mixing coefficient, and fluid viscosity, respectively, then there is usually an evolution equation for the phase-field variable ϕ. We consider the following no-flux or no-flow boundary and initial conditions of (1.1):

$$\begin{aligned} &\frac{\partial \phi}{\partial \mathbf{n}} = \frac{\partial \mu}{\partial \mathbf{n}} = 0, \quad \mathbf{u} = 0 && \text{on } \partial\Omega \times (0, T], \\ &\phi(\mathbf{x}, 0) = \phi^0, \qquad \mathbf{u}(\mathbf{x}, 0) = \mathbf{u}^0 && \text{in } \Omega, \end{aligned} \tag{1.2}$$

where Ω is a bounded domain in $\mathbb{R}^2$ with boundary $\partial\Omega$. The unknowns are the phase function ϕ and the chemical potential μ, the velocity $\mathbf{u}$, the pressure p, and $\mathbf{n}$ denotes the unit outward normal vector on $\partial\Omega$. We can obtain the energy dissipation law in above system (1.1) as follows:

$$\frac{dE(\phi, \mathbf{u})}{dt} = -M\|\nabla\mu\|^2 - \nu\|\nabla\mathbf{u}\|^2, \tag{1.3}$$

where

$$E(\phi, \mathbf{u}) = \int_\Omega \left\{ \frac{1}{2}|\mathbf{u}|^2 + \frac{\lambda}{2}|\nabla\phi|^2 + \lambda G(\phi) \right\} dx$$

is the total energy, and $\|\cdot\|$ is defined as L^2 norm.

For the CHNS system, the equations of the phase-field model are derived from the energy function, so the numerical discretization scheme must satisfy the energy stability. Moreover, the system (1.1) is a coupling of the Cahn-Hilliard [2] and Navier-Stokes equations [15], and in order to reduce the computational effort, it is desirable for the discretization scheme to achieve decoupling of these two equations. Therefore, in the past decades, many numerical schemes have emerged for CHNS systems to satisfy the need for energy stabilization or decoupling. The numerical methods [4–7, 9, 10, 12, 13, 16–18, 23, 25, 29, 32, 34] are mainly the linear stabilization method, the convex splitting method, the invariant energy quadratization method (IEQ) and its variant version, and the scalar auxiliary variable (SAV) method. Feng [10] proposed and analyzed a fully discrete mixed finite element method for the CHNS phase-field model, where the time discretization used an implicit Euler scheme. Kay *et al.* [18] presented a finite element discretization of the variable density CHNS system. Subsequently, they constructed semi-discrete and practical fully-discrete finite element approximation schemes for the CHNS system and analyzed the convergence of the fully-discrete approximation [17]. He *et al.* [14] used the finite element spatial approximation with the time discretization by operator-splitting and a least-squares/conjugate gradient method for the Cahn-Hilliard equation. Shen and Yang [29] constructed several effective time discretization schemes for the coupled nonlinear Cahn-Hilliard two-phase incompressible flows with the matched density case and the variable density case and established the dissipation energy law. Bao *et al.* [1] submitted the semi-implicit