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Stability and Error Analysis of SAV Semi-Discrete Scheme for Cahn-Hilliard-Navier-Stokes Model

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Abstract. We construct first- and second-order time semi-discretization numerical schemes for the Cahn-Hilliard-Navier-Stokes model. This discretization scheme is based on the energy form of the scalar auxiliary variable approach for the coupling terms of model and pressure correction in the Navier-Stokes equations, which are fully decoupled. Then, we apply the fully explicit forms and the two scalar auxiliary variables to obtain stable unconditional energy over time. At the same time, we present the error analysis for the first-order scheme and the convergence rate for all relevant variables in different norms. Finally, numerical examples are presented to validate the theoretical analysis.

AMS subject classifications: 35Q30, 65M12, 65M60, 65P40

Key words: Cahn-Hilliard-Navier-Stokes, fully decoupled, scalar auxiliary variable (SAV), energy stability, error estimates.

1. Introduction

In this article, we consider the Cahn-Hilliard-Navier-Stokes (CHNS) system as follows:

$$\begin{aligned}
&\frac{\partial \phi}{\partial t} + (\mathbf{u}\cdot\nabla)\phi - M\Delta\mu = 0 && \text{in } \Omega\times(0,T],\\
&\mu + \lambda\Delta\phi - \lambda G'(\phi) = 0 && \text{in } \Omega\times(0,T],\\
&\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p - \mu\nabla\phi = 0 && \text{in } \Omega\times(0,T],\\
&\nabla\cdot\mathbf{u} = 0 && \text{in } \Omega\times(0,T],
\end{aligned} \tag{1.1}$$

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where

$$G(\phi)=\frac{1}{4\epsilon^2}(\phi^2-1)^2$$

is a nonlinear free energy density, where ϵ denotes the interface width and $M,\lambda,\nu>0$, describes the mobility, mixing coefficient, and fluid viscosity, respectively, then there is usually an evolution equation for the phase-field variable ϕ. We consider the following no-flux or no-flow boundary and initial conditions of (1.1):

$$\begin{aligned}&\frac{\partial\phi}{\partial\mathbf{n}}=\frac{\partial\mu}{\partial\mathbf{n}}=0, \quad \mathbf{u}=0 && \text{on } \ \partial\Omega\times(0,T],\\ &\phi(\mathbf{x},0)=\phi^0, \qquad \mathbf{u}(\mathbf{x},0)=\mathbf{u}^0 && \text{in } \ \Omega,\end{aligned} \tag{1.2}$$

where Ω is a bounded domain in $\mathbb{R}^2$ with boundary $\partial\Omega$. The unknowns are the phase function ϕ and the chemical potential μ, the velocity $\mathbf{u}$, the pressure p, and $\mathbf{n}$ denotes the unit outward normal vector on $\partial\Omega$. We can obtain the energy dissipation law in above system (1.1) as follows:

$$\frac{dE(\phi,\mathbf{u})}{dt}=-M\|\nabla\mu\|^2-\nu\|\nabla\mathbf{u}\|^2, \tag{1.3}$$

where

$$E(\phi,\mathbf{u})=\int_\Omega\left\{\frac{1}{2}|\mathbf{u}|^2+\frac{\lambda}{2}|\nabla\phi|^2+\lambda G(\phi)\right\}dx$$

is the total energy, and $\|\cdot\|$ is defined as L^2 norm.

For the CHNS system, the equations of the phase-field model are derived from the energy function, so the numerical discretization scheme must satisfy the energy stability. Moreover, the system (1.1) is a coupling of the Cahn-Hilliard [2] and Navier-Stokes equations [15], and in order to reduce the computational effort, it is desirable for the discretization scheme to achieve decoupling of these two equations. Therefore, in the past decades, many numerical schemes have emerged for CHNS systems to satisfy the need for energy stabilization or decoupling. The numerical methods [4–7, 9, 10, 12, 13, 16–18, 23, 25, 29, 32, 34] are mainly the linear stabilization method, the convex splitting method, the invariant energy quadratization method (IEQ) and its variant version, and the scalar auxiliary variable (SAV) method. Feng [10] proposed and analyzed a fully discrete mixed finite element method for the CHNS phase-field model, where the time discretization used an implicit Euler scheme. Kay *et al.* [18] presented a finite element discretization of the variable density CHNS system. Subsequently, they constructed semi-discrete and practical fully-discrete finite element approximation schemes for the CHNS system and analyzed the convergence of the fully-discrete approximation [17]. He *et al.* [14] used the finite element spatial approximation with the time discretization by operator-splitting and a least-squares/conjugate gradient method for the Cahn-Hilliard equation. Shen and Yang [29] constructed several effective time discretization schemes for the coupled nonlinear Cahn-Hilliard two-phase incompressible flows with the matched density case and the variable density case and established the dissipation energy law. Bao *et al.* [1] submitted the semi-implicit finite element method for the moving contact line problem with coupled Cahn-Hilliard and Navier-Stokes equations with generalized Navier boundary conditions. Guo *et al.* [12] designed a C^0 finite element method scheme to preserve the energy law at the discrete level for the variable density two-phase flow model. Han *et al.* [13] suggested the second-order in time scheme for the Cahn-Hilliard-Navier-Stokes phase field model with matched density. The scheme combines the second-order convex splitting and pressure projection methods for the Navier-Stokes equations. Diegel *et al.* [8] proposed the mixed finite element method based on convex splitting for a modified Cahn-Hilliard equation coupled with a nonsteady Darcy-Stokes. Then, to preserve the energy stability natural to the Cahn-Hilliard equation, Diegel *et al.* [9] introduced second-order Adams-Bashforth extrapolations and the trapezoidal norm to present the second-order in time mixed finite element scheme, which combines a standard second-order Crank-Nicolson method for the Navier-Stokes equations and a modification to the Crank-Nicolson method for the Cahn-Hilliard equation. Chen *et al.* [7] constructed adaptive mesh, adaptive time and a nonlinear multigrid finite difference method. Cai *et al.* [3, 4] analyzed the error estimates of semi-discrete and fully discrete energy stabilization schemes for two-phase incompressible flows.

Recently, Yang *et al.* [34, 35] introduced the IEQ method so that a new variable can modify the energy in the original system. The advantage of the IEQ method is that it can explicitly handle nonlinear terms in the discretization process. Subsequently, Shen *et al.* [26–28] proposed the scalar auxiliary variables method, in which an auxiliary variable is introduced and defined as the square of the integral of the potential energy. Lin *et al.* [24] proposed the scheme for approximating the incompressible Navier-Stokes equations based on an auxiliary variable associated with the modified system energy, and the above scheme is unconditionally energy stable with a modified energy. Using a Lagrange multiplier approach, Jia *et al.* [16] used implicit–explicit treatments and constructed the decoupled and unconditionally energy-stabilized time discretization scheme for the two-phase incompressible flow CHNS model. Li *et al.* [20, 21] proposed the scalar auxiliary variable approach in time for the CHNS model. Then, the multiple SAV methods in the first and second-order time discretization for the CHNS system are proposed in [22].

The main contribution of this article is to construct the first-order and second-order schemes for the CHNS equation based on the two scalar auxiliary variables (SAVs) associated with the modified system energy in the time. The schemes are fully decoupled and satisfy the energy dissipation law in the time semi-discretization. Meanwhile, we prove the error estimation and convergence rate in the first-order time semi-discretization. Finally, we provide some numerical examples to validate the theory result.

The rest of this paper is organized as follows. In Section 2, we describe our notations and our main theoretical results. In Section 3, we introduce the first- and second-order SAVs approaches in the time. In Section 4, we prove the error estimates of the first-order SAVs scheme. Numerical experiments are presented in Section 5 to validate our theoretical analysis.

2. Preliminaries

Firstly, we introduce some basic notations. Let $L^m(\Omega)$ be the Banach space with the norm

$$\|v\|_{L^m(\Omega)} = \left(\int_\Omega |v|^m d\Omega\right)^{1/m}, \quad \|v\|_\infty = \|v\|_{L^\infty} = \max_{0\leqslant i\leqslant N} |v_i|, \tag{2.1}$$

and

$$(f,g) = (f,g)_{L^2(\Omega)} = \int_\Omega fg\ d\Omega, \tag{2.2}$$

where $(\cdot,\cdot)$ is the inner products of L^2 and $\mathbf{L}^2$. It is shown that $\|\cdot\|$ be the norms of L^2 and $\mathbf{L}^2$. Nextly, we denote by $W^{k,p}(\Omega)$ the Sobolev space of functions defined on Ω

$$W^{k,p}(\Omega) = \left\{ g : \|g\|_{W_p^k(\Omega)} = \left(\sum_{|\alpha|\le k} \|D^\alpha g\|^p_{L^p(\Omega)}\right)^{1/p} < \infty \right\}, \tag{2.3}$$

in the case $1 \le p < \infty$, and in the case $p = \infty$, $\|g\|_{W^{k,\infty}(\Omega)} = \max_{|\alpha|\le k} \|D^\alpha g\|_{L^\infty(\Omega)}$. To be simple, we set $H^k(\Omega) = W^{k,2}(\Omega)$, and $\|f\|_k = \|f\|_{H^k(\Omega)}$, and $\mathbf{H}^k(\Omega) = [H^k(\Omega)]^d$. Thus, we define

$$\mathbf{H}_0^1(\Omega) = \left\{ \boldsymbol{v} \in \mathbf{H}^1(\Omega) : \boldsymbol{v}|_{\partial\Omega} = 0 \right\}, \tag{2.4}$$

and

$$\begin{aligned} \mathbf{H} &= \left\{ \boldsymbol{v} \in \mathbf{L}^2(\Omega) : \nabla\cdot\boldsymbol{v} = 0 \right\}, \\ \mathbf{V} &= \left\{ \boldsymbol{v} \in \mathbf{H}_0^1(\Omega) : \nabla\cdot\boldsymbol{v} = 0 \right\}, \end{aligned} \tag{2.5}$$

and the trilinear form $b(\cdot,\cdot,\cdot)$ by

$$b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) = \frac{1}{2}\int_\Omega \big((\boldsymbol{u}\cdot\nabla)\boldsymbol{v}\cdot\boldsymbol{w} - (\boldsymbol{u}\cdot\nabla)\boldsymbol{w}\cdot\boldsymbol{v}\big)\ d\boldsymbol{x}. \tag{2.6}$$

It is obvious that the trilinear form $b(\cdot,\cdot,\cdot)$ is skew-symmetric, i.e.

$$b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) = -b(\boldsymbol{u},\boldsymbol{w},\boldsymbol{v}), \quad \forall \boldsymbol{u}\in\mathbf{H}, \quad \boldsymbol{v},\boldsymbol{w}\in\mathbf{H}_0^1(\Omega), \tag{2.7}$$

$$b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{v}) = 0, \quad \forall \boldsymbol{u}\in\mathbf{H}, \quad \boldsymbol{v}\in\mathbf{H}_0^1(\Omega). \tag{2.8}$$

Then, we arrive at the result by using Poincaré inequality

$$\|\boldsymbol{v}\| \le c_1\|\nabla\boldsymbol{v}\|, \quad \forall \boldsymbol{v}\in\mathbf{H}_0^1(\Omega), \tag{2.9}$$

where c_1 is a positive constant depending only on Ω. Finally, we obtain the results by using the Hölder inequality and the Sobolev inequalities [31]

$$\begin{aligned} b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) &\le c\|\nabla\boldsymbol{u}\|^{1/2}\|\boldsymbol{u}\|^{1/2}\|\boldsymbol{v}\|_2^{1/2}\|\nabla\boldsymbol{v}\|^{1/2}\|\boldsymbol{w}\|, \quad d=2, \\ b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) &\le c\|\boldsymbol{u}\|_1\|\boldsymbol{v}\|_1^{1/2}\|\boldsymbol{v}\|_2^{1/2}\|\boldsymbol{w}\|, \quad d=3. \end{aligned} \tag{2.10}$$

We also frequently use the following inequalities [31]:

$$b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) \leq \begin{cases} c\|\boldsymbol{u}\|_1\|\boldsymbol{v}\|_1\|\boldsymbol{w}\|_1, \\ c\|\boldsymbol{u}\|_2\|\boldsymbol{v}\|_0\|\boldsymbol{w}\|_1, \\ c\|\boldsymbol{u}\|_2\|\boldsymbol{v}\|_1\|\boldsymbol{w}\|_0, \quad d \leq 4, \\ c\|\boldsymbol{u}\|_1\|\boldsymbol{v}\|_2\|\boldsymbol{w}\|_0, \\ c\|\boldsymbol{u}\|_0\|\boldsymbol{v}\|_2\|\boldsymbol{w}\|_1, \end{cases} \tag{2.11}$$

where c is a positive constant depending only on Ω. Let $P_{\mathbf{H}}u$ be the projection operator in $L^2(\Omega)$ onto $\mathbf{H}$. We refer to [30, (1.47)], which follows

$$\|P_{\mathbf{H}}u\|_1 \leq C(\Omega)\|u\|_1, \quad \forall u \in \mathbf{H}^1(\Omega). \tag{2.12}$$

We will frequently use the following discrete version of the Grönwall lemma [15].

Lemma 2.1. *Let* $a_k, b_k, c_k, d_k, \gamma_k, \tau_k$ *be nonnegative real numbers such that*

$$a_{k+1} - a_k + b_{k+1}\tau_{k+1} + c_{k+1}\tau_{k+1} - c_k\tau_k \leq a_k d_k \tau_k + \gamma_{k+1}\tau_{k+1} \tag{2.13}$$

for all $0 \leq k \leq m$. *Then*

$$a_{m+1} + \sum_{k=0}^{m+1} b_k\tau_k \leq \exp\left(\sum_{k=0}^{m} d_k\tau_k\right)\left\{a_0 + (b_0 + c_0)\,\tau_0 + \sum_{k=1}^{m+1}\gamma_k\tau_k\right\}. \tag{2.14}$$

Throughout the paper, we use C, with or without subscript, to denote a positive constant independent of discretization parameters, which could have uncertainty values at different places.

3. The SAVs approach

3.1. SAVs reformulation

Let $\gamma > 0$ be a positive constant, $F(\phi) = G(\phi) - \gamma\phi^2/2$, $E_1(\phi) = \int_\Omega F(\phi)d\mathbf{x}$ and $E_2(\mathbf{u}) = \|\mathbf{u}\|^2/2$, where the term $\gamma\phi^2/2$ is to simply the analysis [26]. We introduce two auxiliary variables

$$r(t) = \sqrt{E_1(\phi) + C_1}, \quad C_1 > \gamma, \tag{3.1}$$

$$q(t) = \sqrt{E_2(\mathbf{u}) + C_2}, \quad C_2 > 0. \tag{3.2}$$

Then

$$2q\frac{dq}{dt} = \int_\Omega \frac{\partial \mathbf{u}}{\partial t}\cdot\mathbf{u} = \int_\Omega\left(\frac{\partial \mathbf{u}}{\partial t} + 2\mathbf{u}\cdot\nabla\mathbf{u}\right)\cdot\mathbf{u} - \int_{\partial\Omega}(\mathbf{n}\cdot\mathbf{u})\frac{1}{2}|\mathbf{u}|^2, \tag{3.3}$$

where $\mathbf{n}$ is the outward-pointing unit vector normal to the boundary $\partial\Omega$, and we have used integration by part, the Eq. (1.2), and the divergence theorem. It should be emphasized that both $q(t)$ and $E_2(t)$ are scalar variables, not field functions. At $t=0$,

$$q(0)=\left(\frac{1}{2}\int_\Omega |\mathbf{u}|^2+C_2\right)^{\frac{1}{2}}, \tag{3.4}$$

and reformulate the Eq. (1.1) as

$$\begin{aligned}
&\frac{\partial\phi}{\partial t}+\frac{r}{\sqrt{E_1(\phi)+C_1}}(\mathbf{u}\cdot\nabla)\phi-M\Delta\mu=0 && \text{in } \Omega\times(0,T], && \text{(3.5a)}\\
&\mu+\lambda\Delta\phi-\lambda\gamma\phi-\lambda\frac{r}{\sqrt{E_1(\phi)+C_1}}F'(\phi)=0 && \text{in } \Omega\times(0,T], && \text{(3.5b)}\\
&\frac{dr}{dt}-\frac{1}{2\sqrt{E_1(\phi)+C_1}}\int_\Omega F'(\phi)\frac{\partial\phi}{\partial t}d\mathbf{x}=0 && \text{in } \Omega\times(0,T], && \text{(3.5c)}\\
&\frac{\partial\mathbf{u}}{\partial t}+\frac{q}{\sqrt{E_2(\mathbf{u})+C_2}}\mathbf{u}\cdot\nabla\mathbf{u}-\nu\Delta\mathbf{u}+\nabla p && && \\
&\qquad-\frac{r}{\sqrt{E_1(\phi)+C_1}}\mu\nabla\phi=0 && \text{in } \Omega\times(0,T], && \text{(3.5d)}\\
&\nabla\cdot\mathbf{u}=0 && \text{in } \Omega\times(0,T], && \text{(3.5e)}\\
&2q\frac{\partial q}{\partial t}-\mathbf{u}\frac{\partial\mathbf{u}}{\partial t}-\frac{2q}{\sqrt{E_2(\mathbf{u})+C_2}}\int_\Omega \mathbf{u}\cdot\nabla\mathbf{u}\cdot\mathbf{u}\,d\mathbf{x}=0 && \text{in } \Omega\times(0,T]. && \text{(3.5f)}
\end{aligned}$$

Thus, we have the dissipation law

$$\frac{d\tilde{E}(\phi,\mathbf{u},r,q)}{dt}=-M\|\nabla\mu\|^2-\nu\|\nabla\mathbf{u}\|^2, \tag{3.6}$$

where

$$\tilde{E}(\phi,\mathbf{u},r,q)=\int_\Omega\frac{1}{2}\left(|\mathbf{u}|^2+\lambda\gamma|\phi|^2+\lambda|\nabla\phi|^2\right)d\mathbf{x}+q^2+2\lambda r^2.$$

3.2. A first-order semi-discrete scheme

We denote

$$\tau=\frac{T}{N},\quad t^n=n\tau,\quad \delta_\tau f^n=\frac{f^n-f^{n-1}}{\tau}\quad \text{for}\quad n\le N.$$

This is first-order scheme as follows: Find $(\phi^{n+1},\ \mu^{n+1},\ \tilde{\mathbf{u}}^{n+1},\ \mathbf{u}^{n+1},\ p^{n+1},\ r^{n+1},\ q^{n+1})$ such that

$$\frac{\phi^{n+1}-\phi^n}{\tau}+\frac{r^{n+1}}{\sqrt{E_1(\phi^n)+C_1}}(\mathbf{u}^n\cdot\nabla)\phi^n-M\Delta\mu^{n+1}=0, \tag{3.7}$$

$$\mu^{n+1}+\lambda\Delta\phi^{n+1}-\lambda\gamma\phi^{n+1}-\lambda\frac{r^{n+1}}{\sqrt{E_1(\phi^n)+C_1}}F'(\phi^n)=0, \tag{3.8}$$

$$\begin{aligned}&\frac{r^{n+1}-r^n}{\tau}-\frac{1}{2\sqrt{E_1(\phi^n)+C_1}}\\&\quad\times\left(\left(F'(\phi^n),\frac{\phi^{n+1}-\phi^n}{\tau}\right)+\frac{1}{\lambda}\left(\mu^{n+1},\mathbf{u}^n\cdot\nabla\phi^n\right)-\frac{1}{\lambda}\left(\tilde{\mathbf{u}}^{n+1},\mu^n\nabla\phi^n\right)\right)=0,\end{aligned} \tag{3.9}$$

$$\begin{aligned}&\frac{\tilde{\mathbf{u}}^{n+1}-\mathbf{u}^n}{\tau}+\frac{q^{n+1}}{\sqrt{E_2(\mathbf{u}^n)+C_2}}\mathbf{u}^n\cdot\nabla\mathbf{u}^n-\nu\Delta\tilde{\mathbf{u}}^{n+1}+\nabla p^n\\&\quad-\frac{r^{n+1}}{\sqrt{E_1(\phi^n)+C_1}}\mu^n\nabla\phi^n=0,\end{aligned} \tag{3.10}$$

$$\tilde{\mathbf{u}}^{n+1}|_{\partial\Omega}=0, \tag{3.11}$$

$$\nabla\cdot\mathbf{u}^{n+1}=0,\quad \mathbf{u}^{n+1}\cdot\mathbf{n}|_{\partial\Omega}=0, \tag{3.12}$$

$$\frac{\mathbf{u}^{n+1}-\tilde{\mathbf{u}}^{n+1}}{\tau}+\nabla\left(p^{n+1}-p^n\right)=0, \tag{3.13}$$

$$2q^{n+1}\frac{q^{n+1}-q^n}{\tau}-\left(\frac{\tilde{\mathbf{u}}^{n+1}-\mathbf{u}^n}{\tau}+\frac{2q^{n+1}}{\sqrt{E_2(\mathbf{u}^n)+C_2}}\mathbf{u}^n\cdot\nabla\mathbf{u}^n,\tilde{\mathbf{u}}^{n+1}\right)=0. \tag{3.14}$$

Note that we added the terms $(1/\lambda)((\mu^{n+1},\mathbf{u}^n\cdot\nabla\phi^n)-(\tilde{\mathbf{u}}^{n+1},\mu^n\cdot\nabla\phi^n))$ in Eq. (3.9), which is a first-order approximation to $(\mu,\ \mathbf{u}\nabla\phi)-(\mathbf{u},\mu\nabla\phi)=0$. For continuous case (3.5d)-(3.5e), we use the first-order pressure-correction scheme [11] discretization to obtain (3.10)-(3.13).

Remark 3.1. We note that the (3.14) is a nonlinear quadratic equation for q^{n+1}, and rewrite it in the following form:

$$2(q^{n+1})^2-2\left(q^n+\frac{\tau\left(\mathbf{u}^n\cdot\nabla\mathbf{u}^n,\tilde{\mathbf{u}}^{n+1}\right)}{\sqrt{E_2(\mathbf{u}^n)+C_2}}\right)q^{n+1}-\left(\tilde{\mathbf{u}}^{n+1}-\mathbf{u}^n,\tilde{\mathbf{u}}^{n+1}\right)=0. \tag{3.15}$$

The above equation can be simplified as

$$ax^2+bx+c=0, \tag{3.16}$$

where the coefficients are

$$a=2,\quad b=-2\left(q^n+\frac{\tau\left(\mathbf{u}^n\cdot\nabla\mathbf{u}^n,\tilde{\mathbf{u}}^{n+1}\right)}{\sqrt{E_2(\mathbf{u}^n)+C_2}}\right),\quad c=-\left(\tilde{\mathbf{u}}^{n+1}-\mathbf{u}^n,\tilde{\mathbf{u}}^{n+1}\right). \tag{3.17}$$

According to $-2ab\geq-(a^2+b^2)$, we can obtain

$$b^2-4ac=4\left|\left(q^n+\frac{\tau\left(\mathbf{u}^n\cdot\nabla\mathbf{u}^n,\tilde{\mathbf{u}}^{n+1}\right)}{\sqrt{E_2(\mathbf{u}^n)+C_2}}\right)\right|^2+8\left(\|\tilde{\mathbf{u}}^{n+1}\|^2-\tilde{\mathbf{u}}^{n+1}\mathbf{u}^n\right)$$

$$
\begin{aligned}
&= 4\left|\left(q^n + \frac{\tau\left(\mathbf{u}^n\cdot\nabla\mathbf{u}^n, \tilde{\mathbf{u}}^{n+1}\right)}{\sqrt{E_2(\mathbf{u}^n)+C_2}}\right)\right|^2 \\
&\quad + 4\left(\|\tilde{\mathbf{u}}^{n+1}\|^2 - \|\mathbf{u}^n\|^2 + \|\tilde{\mathbf{u}}^{n+1}-\mathbf{u}^n\|^2\right) \\
&= 4\left|\left(q^n + \frac{\tau\left(\mathbf{u}^n\cdot\nabla\mathbf{u}^n, \tilde{\mathbf{u}}^{n+1}\right)}{\sqrt{E_2(\mathbf{u}^n)+C_2}}\right)\right|^2 \\
&\quad + 4\left(\|\tilde{\mathbf{u}}^{n+1}\|^2 - \|\tilde{\mathbf{u}}^{n+1} - \left(\tilde{\mathbf{u}}^{n+1}-\mathbf{u}^n\right)\|^2 + \|\tilde{\mathbf{u}}^{n+1}-\mathbf{u}^n\|^2\right) \\
&\geq 4\left|\left(q^n + \frac{\tau\left(\mathbf{u}^n\cdot\nabla\mathbf{u}^n, \tilde{\mathbf{u}}^{n+1}\right)}{\sqrt{E_2(\mathbf{u}^n)+C_2}}\right)\right|^2 \\
&\quad + 4\left(\|\tilde{\mathbf{u}}^{n+1}\|^2 - \|\tilde{\mathbf{u}}^{n+1}\|^2 - \|\tilde{\mathbf{u}}^{n+1}-\mathbf{u}^n\|^2 + \|\tilde{\mathbf{u}}^{n+1}-\mathbf{u}^n\|^2\right) \\
&\geq 4\left|\left(q^n + \frac{\tau\left(\mathbf{u}^n\cdot\nabla\mathbf{u}^n, \tilde{\mathbf{u}}^{n+1}\right)}{\sqrt{E_2(\mathbf{u}^n)+C_2}}\right)\right|^2 \geq 0. \qquad (3.18)
\end{aligned}
$$

Thus, we obtain that the Eq. (3.14) has real number solutions.

Due to the Remark 3.1, the nonlinear quadratic equation (3.14) has two solutions. Therefore, we need to give how to choose a more desirable root. A similar approach has been given in [19, 20, 24] and we omit the detailed process here. Since the exact solution of $q^{n+1}/\sqrt{E_2(\mathbf{u}^{n+1})+C_2}$ is 1, we choose the root q^{n+1} such that $q^{n+1}/\sqrt{E_2(\mathbf{u}^{n+1})+C_2}$ is closer to 1.

3.3. Energy stability

Next, we prove that the scheme (3.7)-(3.14) is unconditionally energy stable.

Theorem 3.1. *The scheme* (3.7)-(3.14) *is unconditional energy stable in the sense that*

$$
\tilde{E}^{n+1}(\phi,\mathbf{u},r,q) - \tilde{E}^{n}(\phi,\mathbf{u},r,q) \leq -2M\tau\|\nabla\mu^{n+1}\|^2 - 2\tau\nu\|\nabla\tilde{\mathbf{u}}^{n+1}\|^2, \qquad (3.19)
$$

where

$$
\begin{aligned}
\tilde{E}^{n+1}(\phi,\mathbf{u},r,q) &= \lambda\|\nabla\phi^{n+1}\|^2 + \lambda\gamma\|\phi^{n+1}\|^2 + \frac{1}{2}\|\mathbf{u}^{n+1}\|^2 \\
&\quad + \frac{\tau^2}{2}\|\nabla p^{n+1}\|^2 + 2\lambda|r^{n+1}|^2 + |q^{n+1}|^2. \qquad (3.20)
\end{aligned}
$$

Proof. We prove the unconditionally energy stability. Taking the inner products of (3.7) with $2\tau\mu^{n+1}$, (3.8) with $2(\phi^{n+1}-\phi^n)$, respectively, and multiplying (3.9) with $4\lambda\tau r^{n+1}$, we can obtain

$$
\left(\frac{\phi^{n+1}-\phi^n}{\tau} + \frac{r^{n+1}}{\sqrt{E_1(\phi^n)+C_1}}(\mathbf{u}^n\cdot\nabla)\phi^n, 2\tau\mu^{n+1}\right) = \left(M\Delta\mu^{n+1}, 2\tau\mu^{n+1}\right), \qquad (3.21)
$$

and

$$
\begin{aligned}
&\left(\mu^{n+1}, 2(\phi^{n+1}-\phi^n)\right)\\
=&\left(-\lambda\Delta\phi^{n+1}+\lambda\gamma\phi^{n+1}+\frac{\lambda r^{n+1}}{\sqrt{E_1(\phi^n)+C_1}}F'(\phi^n), 2(\phi^{n+1}-\phi^n)\right),
\end{aligned} \tag{3.22}
$$

$$
\begin{aligned}
&\left(\frac{r^{n+1}-r^n}{\tau}, 4\lambda\tau r^{n+1}\right)\\
=&\frac{4\lambda\tau r^{n+1}}{2\sqrt{E_1(\phi^n)+C_1}}\left(\left(F'(\phi^n), \frac{\phi^{n+1}-\phi^n}{\tau}\right)\right)\\
&+\frac{4\lambda\tau r^{n+1}}{2\sqrt{E_1(\phi^n)+C_1}}\left(\frac{1}{\lambda}\left(\mu^{n+1}, \mathbf{u}^n\cdot\nabla\phi^n\right)-\frac{1}{\lambda}\left(\tilde{\mathbf{u}}^{n+1}, \mu^n\nabla\phi^n\right)\right).
\end{aligned} \tag{3.23}
$$

Thus, this leads to the following equation:

$$
\begin{aligned}
&\lambda\left(\|\nabla\phi^{n+1}\|^2-\|\nabla\phi^n\|^2+\|\nabla\phi^{n+1}-\nabla\phi^n\|^2\right)\\
&\quad+\lambda\gamma\left(\|\phi^{n+1}\|^2-\|\phi^n\|^2+\|\phi^{n+1}-\phi^n\|^2\right)\\
&\quad+2\lambda\left(|r^{n+1}|^2-|r^n|^2+|r^{n+1}-r^n|^2\right)\\
&\quad+\frac{2\tau r^{n+1}}{\sqrt{E_1(\phi^n)+C_1}}\left(\tilde{\mathbf{u}}^{n+1}, \mu^n\nabla\phi^n\right)\\
=&-2M\tau\|\nabla\mu^{n+1}\|^2.
\end{aligned} \tag{3.24}
$$

Taking the inner product of (3.10) with $2\tau\tilde{\mathbf{u}}^{n+1}$ leads to

$$
\begin{aligned}
&\left(\frac{\tilde{\mathbf{u}}^{n+1}-\mathbf{u}^n}{\tau}+\frac{q^{n+1}}{\sqrt{E_2(\mathbf{u}^n)+C_2}}\mathbf{u}^n\cdot\nabla\mathbf{u}^n-\nu\Delta\tilde{\mathbf{u}}^{n+1}+\nabla p, 2\tau\tilde{\mathbf{u}}^{n+1}\right)\\
=&\left(\frac{r^{n+1}}{\sqrt{E_1(\phi^n)+C_1}}\mu^n\nabla\phi^n, 2\tau\tilde{\mathbf{u}}^{n+1}\right),
\end{aligned} \tag{3.25}
$$

and

$$
\begin{aligned}
&\|\tilde{\mathbf{u}}^{n+1}\|^2-\|\mathbf{u}^n\|^2+\|\tilde{\mathbf{u}}^{n+1}-\mathbf{u}^n\|^2\\
&\quad+\frac{2\tau q^{n+1}}{\sqrt{E_2(\mathbf{u}^n)+C_2}}\left(\mathbf{u}^n\cdot\nabla\mathbf{u}^n, \tilde{\mathbf{u}}^{n+1}\right)+2\tau\left(\nabla p^n, \tilde{\mathbf{u}}^{n+1}\right)\\
=&\left(\frac{2\tau r^{n+1}}{\sqrt{E_1(\phi^n)+C_1}}\mu^n\nabla\phi^n, \tilde{\mathbf{u}}^{n+1}\right)-2\nu\tau\|\nabla\tilde{\mathbf{u}}^{n+1}\|^2,
\end{aligned} \tag{3.26}
$$

$$
\begin{aligned}
&\|\tilde{\mathbf{u}}^{n+1}\|^2-\|\mathbf{u}^n\|^2+\|\tilde{\mathbf{u}}^{n+1}-\mathbf{u}^n\|^2\\
&\quad+\frac{2\tau q^{n+1}}{\sqrt{E_2(\mathbf{u}^n)+C_2}}\left(\mathbf{u}^n\cdot\nabla\mathbf{u}^n, \tilde{\mathbf{u}}^{n+1}\right)\\
=&\left(\frac{2\tau r^{n+1}}{\sqrt{E_1(\phi^n)+C_1}}\mu^n\nabla\phi^n, \tilde{\mathbf{u}}^{n+1}\right).
\end{aligned} \tag{3.27}
$$

Recalling (3.14), we have

$$\begin{aligned}
&|q^{n+1}|^2 - |q^n|^2 + |q^{n+1} - q^n|^2 \\
=& \left(\left(\tilde{\mathbf{u}}^{n+1} - \mathbf{u}^n\right) + \frac{2\tau q^{n+1}}{\sqrt{E_2(\mathbf{u}^n) + C_2}} \mathbf{u}^n \cdot \nabla \mathbf{u}^n, \tilde{\mathbf{u}}^{n+1} \right).
\end{aligned} \tag{3.28}$$

For the formula (3.13), we get

$$\mathbf{u}^{n+1} + \tau \nabla p^{n+1} = \tilde{\mathbf{u}}^{n+1} + \tau \nabla p^n. \tag{3.29}$$

Taking the inner product of (3.29) with itself on both sides and noticing that

$$\left(\nabla p^{n+1}, \mathbf{u}^{n+1}\right) = -\left(p^{n+1}, \nabla \cdot \mathbf{u}^{n+1}\right) = 0,$$

we have

$$\|\mathbf{u}^{n+1}\|^2 + (\tau)^2 \|\nabla p^{n+1}\|^2 = \|\tilde{\mathbf{u}}^{n+1}\|^2 + 2\tau(\nabla p^n, \tilde{\mathbf{u}}^{n+1}) + (\tau)^2 \|\nabla p^n\|^2, \tag{3.30}$$

and then

$$\frac{1}{2}\|\mathbf{u}^{n+1}\|^2 + \frac{1}{2}(\tau)^2 \|\nabla p^{n+1}\|^2 = \frac{1}{2}\|\tilde{\mathbf{u}}^{n+1}\|^2 + \frac{1}{2}(\tau)^2 \|\nabla p^n\|^2. \tag{3.31}$$

Combining (3.27) with (3.28) and (3.30) results in

$$\begin{aligned}
&\frac{1}{2}\left(\|\mathbf{u}^{n+1}\|^2 - \|\mathbf{u}^n\|^2 + \|\tilde{\mathbf{u}}^{n+1} - \mathbf{u}^n\|^2\right) \\
&\quad + |q^{n+1}|^2 - |q^n|^2 + |q^{n+1} - q^n|^2 \\
&\quad + \frac{1}{2}\tau^2 \left(\|\nabla p^{n+1}\|^2 - \|\nabla p^n\|^2\right) \\
=& \left(\frac{2\tau r^{n+1}}{\sqrt{E_1(\phi^n) + C_1}} \mu^n \nabla \phi^n, \tilde{\mathbf{u}}^{n+1} \right) - 2\nu\tau \|\nabla \tilde{\mathbf{u}}^{n+1}\|^2.
\end{aligned} \tag{3.32}$$

Thus, we can obtain the desired result by combining (3.24) with (3.32) as follows:

$$\begin{aligned}
&\lambda\left(\|\nabla\phi^{n+1}\|^2 - \|\nabla\phi^n\|^2 + \|\nabla\phi^{n+1} - \nabla\phi^n\|^2\right) \\
&\quad + \lambda\gamma\left(\|\phi^{n+1}\|^2 - \|\phi^n\|^2 + \|\phi^{n+1} - \phi^n\|^2\right) \\
&\quad + \frac{1}{2}\left(\|\mathbf{u}^{n+1}\|^2 - \|\mathbf{u}^n\|^2 + \|\tilde{\mathbf{u}}^{n+1} - \mathbf{u}^n\|^2\right) \\
&\quad + |q^{n+1}|^2 - |q^n|^2 + |q^{n+1} - q^n|^2 \\
&\quad + \frac{1}{2}\tau^2\left(\|\nabla p^{n+1}\|^2 - \|\nabla p^n\|^2\right) \\
&\quad + 2\lambda\left(|r^{n+1}|^2 - |r^n|^2 + |r^{n+1} - r^n|^2\right) \\
=& -2M\tau\|\nabla\mu^{n+1}\|^2 - 2\nu\tau\|\nabla\tilde{\mathbf{u}}^{n+1}\|^2 \le 0.
\end{aligned} \tag{3.33}$$

This completes the proof. $\square$

3.4. A second-order semi-discrete scheme

We get the second-order SAVs scheme: Find $(\phi^{n+1}, \mu^{n+1}, \tilde{\mathbf{u}}^{n+1}, \mathbf{u}^{n+1}, p^{n+1}, r^{n+1}, q^{n+1})$ such that

$$\frac{3\phi^{n+1}-4\phi^n+\phi^{n-1}}{2\tau}+\frac{r^{n+1}}{\sqrt{E_1(\hat{\phi}^{n+1})+C_1}}\left(\hat{\mathbf{u}}^{n+1}\cdot\nabla\right)\hat{\phi}^{n+1}-M\Delta\mu^{n+1}=0, \tag{3.34}$$

$$\mu^{n+1}+\lambda\Delta\phi^{n+1}-\lambda\gamma\phi^{n+1}-\frac{r^{n+1}}{\sqrt{E_1(\hat{\phi}^{n+1})+C_1}}\left(F'(\hat{\phi}^{n+1})\right)=0, \tag{3.35}$$

$$\begin{aligned}&\frac{3r^{n+1}-4r^n+r^{n-1}}{2\tau}-\frac{1}{2\sqrt{E_1(\hat{\phi}^{n+1})+C_1}}\left(F'(\hat{\phi}^{n+1}),\frac{3\phi^{n+1}-4\phi^n+\phi^{n-1}}{2\tau}\right)\\&\quad-\frac{1}{2\lambda\sqrt{E_1(\hat{\phi}^{n+1})+C_1}}\Big(\left(\mu^{n+1},\hat{\mathbf{u}}^{n+1}\cdot\nabla\hat{\phi}^{n+1}\right)-\left(\tilde{\mathbf{u}}^{n+1},\hat{\mu}^{n+1}\nabla\hat{\phi}^{n+1}\right)\Big)=0,\end{aligned} \tag{3.36}$$

$$\begin{aligned}&\frac{3\tilde{\mathbf{u}}^{n+1}-4\mathbf{u}^n+\mathbf{u}^{n-1}}{2\tau}+\frac{q^{n+1}}{\sqrt{E_2(\hat{\mathbf{u}}^{n+1})+C_2}}\hat{\mathbf{u}}^{n+1}\cdot\nabla\hat{\mathbf{u}}^{n+1}-\nu\Delta\tilde{\mathbf{u}}^{n+1}+\nabla p^n\\&\quad-\frac{r^{n+1}}{\sqrt{E_1(\hat{\phi}^{n+1})+C_1}}\hat{\mu}^{n+1}\nabla\hat{\phi}^{n+1}=0,\end{aligned} \tag{3.37}$$

$$\tilde{\mathbf{u}}^{n+1}|_{\partial\Omega}=0, \tag{3.38}$$

$$\frac{3\mathbf{u}^{n+1}-3\tilde{\mathbf{u}}^{n+1}}{2\tau}+\nabla\left(p^{n+1}-p^n+\nu\nabla\cdot\tilde{\mathbf{u}}^{n+1}\right)=0, \tag{3.39}$$

$$\nabla\cdot\mathbf{u}^{n+1}=0,\quad \mathbf{u}^{n+1}\cdot\mathbf{n}|_{\partial\Omega}=0, \tag{3.40}$$

$$\begin{aligned}&2q^{n+1}\frac{3q^{n+1}-4q^n+q^{n-1}}{2\tau}-\left(\frac{3\mathbf{u}^{n+1}-4\mathbf{u}^n+\mathbf{u}^{n-1}}{2\tau},\mathbf{u}^{n+1}\right)\\&\quad-\left(\frac{2q^{n+1}}{\sqrt{E_2(\hat{\mathbf{u}}^{n+1})+C_2}}\hat{\mathbf{u}}^{n+1}\cdot\nabla\hat{\mathbf{u}}^{n+1},\tilde{\mathbf{u}}^{n+1}\right)=0,\end{aligned} \tag{3.41}$$

where $\hat{\mathbf{u}}^{n+1}=2\mathbf{u}^n-\mathbf{u}^{n-1}$, $\hat{\phi}^{n+1}=2\phi^n-\phi^{n-1}$.

3.5. Energy stability

The second-order semi-discrete scheme is unconditionally energy-stable as we show below.

Theorem 3.2. *Let $g^0=0$, $g^{n+1}=\nu\nabla\cdot\tilde{\mathbf{u}}^{n+1}$, $H^{n+1}=p^{n+1}+g^{n+1}$, $n\geq 0$. The scheme* (3.34)-(3.41) *admits a unique solution, and it is unconditionally energy stable in the sense that*

$$\begin{aligned}&\tilde{E}^{n+1}(\phi,\mathbf{u},r,q)-\tilde{E}^n(\phi,\mathbf{u},r,q)\\&\leq-\nu\tau\|\nabla\times\tilde{\mathbf{u}}^{n+1}\|^2-\nu\tau\|\nabla\tilde{\mathbf{u}}^{n+1}\|^2-2M\tau\left\|\nabla\mu^{n+1}\right\|^2,\end{aligned} \tag{3.42}$$

where

$$
\begin{aligned}
\tilde{E}^{n+1}(\phi,\mathbf{u},r,q) = & \frac{\lambda}{2}\left(\|\nabla\phi^{n+1}\|^2+\|2\phi^{n+1}-\phi^n\|^2\right) \\
& +\frac{\lambda\gamma}{2}\left(\|\phi^{n+1}\|^2+\|2\phi^{n+1}-\phi^n\|^2\right) \\
& +\lambda\left(|r^{n+1}|^2+|2r^{n+1}-r^n|^2\right)+\frac{2}{3}\tau^2\|\nabla H^{n+1}\|^2 \\
& +\frac{\tau}{\nu}\|g^{n+1}\|^2+|q^{n+1}|^2+|2q^{n+1}-q^n|^2.
\end{aligned} \tag{3.43}
$$

Proof. Taking the inner products of (3.34) with $2\tau\mu^{n+1}$, (3.35) with $(3\phi^{n+1}-4\phi^n+\phi^{n-1})$, respectively, and multiplying (3.36) with $4\lambda\tau r^{n+1}$, and using the identity

$$
2a(3a-4b+c)=|a|^2+|2a-b|^2-|b|^2-|2b-c|^2+|a-2b+c|^2, \tag{3.44}
$$

we can obtain

$$
\begin{aligned}
& \frac{\lambda}{2}\left(\|\nabla\phi^{n+1}\|^2+\|2\nabla\phi^{n+1}-\nabla\phi^n\|^2-\|\nabla\phi^n\|^2-\|2\nabla\phi^n-\nabla\phi^{n-1}\|^2\right) \\
& +\frac{\lambda\gamma}{2}\left(\|\phi^{n+1}\|^2+\|2\phi^{n+1}-\phi^n\|^2-\|\phi^n\|^2-\|2\phi^n-\phi^{n-1}\|^2\right) \\
& +\lambda\left(|r^{n+1}|^2+|2r^{n+1}-r^n|^2-|r^n|^2-|2r^n-r^{n-1}|^2\right) \\
& +\frac{\lambda}{2}\|\nabla\phi^{n+1}-2\nabla\phi^n+\nabla\phi^{n-1}\|^2 \\
& +\frac{\lambda\gamma}{2}\|\phi^{n+1}-2\phi^n+\phi^{n-1}\|^2+\lambda|r^{n+1}-2r^n+r^{n-1}|^2 \\
= & -2M\tau\|\nabla\mu^{n+1}\|^2-\frac{2\tau r^{n+1}}{\sqrt{E_1\left(\hat{\phi}^{n+1}\right)+C_1}}\left(\tilde{\mathbf{u}}^{n+1},\hat{\mu}^{n+1}\nabla\hat{\phi}^{n+1}\right).
\end{aligned} \tag{3.45}
$$

Taking the inner product (3.37) with $2\tau\tilde{\mathbf{u}}^{n+1}$ leads to

$$
\begin{aligned}
& \left(3\tilde{\mathbf{u}}^{n+1}-4\mathbf{u}^n+\mathbf{u}^{n+1},\ \tilde{\mathbf{u}}^{n+1}\right) \\
& +\frac{2\tau q^{n+1}}{\sqrt{E_2(\mathbf{u}^n)+C_2}}\left(\hat{\mathbf{u}}^n\cdot\nabla\hat{\mathbf{u}}^n,\ \tilde{\mathbf{u}}^{n+1}\right) \\
& -2\nu\tau\tilde{\mathbf{u}}^{n+1}\Delta\tilde{\mathbf{u}}^{n+1}+2\tau\left(\nabla p^n,\ \tilde{\mathbf{u}}^{n+1}\right) \\
= & \frac{2\tau r^{n+1}}{\sqrt{E_1\left(\hat{\phi}^{n+1}\right)+C_1}}\left(\hat{\mu}^{n+1}\nabla\hat{\phi}^{n+1},\ \tilde{\mathbf{u}}^{n+1}\right).
\end{aligned} \tag{3.46}
$$

Recalling (3.39) and (3.44), the first term on the left-hand side of (3.46) can be transformed into

$$
\begin{aligned}
& \left(3\tilde{\mathbf{u}}^{n+1}-4\mathbf{u}^n+\mathbf{u}^{n-1},\tilde{\mathbf{u}}^{n+1}\right) \\
= & \left(3\left(\tilde{\mathbf{u}}^{n+1}-\mathbf{u}^{n+1}\right)+3\mathbf{u}^{n+1}-4\mathbf{u}^n+\mathbf{u}^{n-1},\tilde{\mathbf{u}}^{n+1}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= 3\left(\tilde{\mathbf{u}}^{n+1}-\mathbf{u}^{n+1},\tilde{\mathbf{u}}^{n+1}\right)+\left(3\mathbf{u}^{n+1}-4\mathbf{u}^{n}+\mathbf{u}^{n-1},\mathbf{u}^{n+1}\right)\\
&\quad+\left(3\mathbf{u}^{n+1}-4\mathbf{u}^{n}+\mathbf{u}^{n-1},\tilde{\mathbf{u}}^{n+1}-\mathbf{u}^{n+1}\right)\\
&=\frac{3}{2}\left(\|\tilde{\mathbf{u}}^{n+1}\|^2-\|\mathbf{u}^{n+1}\|^2+\|\tilde{\mathbf{u}}^{n+1}-\mathbf{u}^{n+1}\|^2\right)\\
&\quad+\frac{1}{2}\|\mathbf{u}^{n+1}\|^2+\frac{1}{2}\|2\mathbf{u}^{n+1}-\mathbf{u}^{n}\|^2-\frac{1}{2}\|\mathbf{u}^{n}\|^2\\
&\quad-\frac{1}{2}\|2\mathbf{u}^{n}-\mathbf{u}^{n-1}\|^2+\frac{1}{2}\|\mathbf{u}^{n+1}-2\mathbf{u}^{n}+\mathbf{u}^{n-1}\|^2.
\end{aligned}
\tag{3.47}
$$

By definition, we rewrite (3.39) as

$$
\sqrt{3}\mathbf{u}^{n+1}+\frac{2}{\sqrt{3}}\tau\nabla H^{n+1}=\sqrt{3}\tilde{\mathbf{u}}^{n+1}+\frac{2}{\sqrt{3}}\tau\nabla H^{n}. \tag{3.48}
$$

Taking the inner product of (3.48) with itself on both sides, we have

$$
\begin{aligned}
&3\|\mathbf{u}^{n+1}\|^2+\frac{4}{3}\tau^2\|\nabla H^{n+1}\|^2\\
&=3\|\tilde{\mathbf{u}}^{n+1}\|^2+\frac{4}{3}\tau^2\|\nabla H^{n}\|^2+4\tau\left(\tilde{\mathbf{u}}^{n+1},\nabla p^{n}\right)+4\tau\left(\tilde{\mathbf{u}}^{n+1},\nabla g^{n}\right),
\end{aligned}
\tag{3.49}
$$

and then

$$
\begin{aligned}
&\frac{3}{2}\|\mathbf{u}^{n+1}\|^2+\frac{2}{3}\tau^2\|\nabla H^{n+1}\|^2\\
&=\frac{3}{2}\|\tilde{\mathbf{u}}^{n+1}\|^2+\frac{2}{3}\tau^2\|\nabla H^{n}\|^2+2\tau\left(\tilde{\mathbf{u}}^{n+1},\nabla p^{n}\right)+2\tau\left(\tilde{\mathbf{u}}^{n+1},\nabla g^{n}\right).
\end{aligned}
\tag{3.50}
$$

The last term on the right-hand side can be controlled by

$$
\begin{aligned}
2\tau\left(\tilde{\mathbf{u}}^{n+1},\nabla g^{n}\right)&=-\frac{2\tau}{\nu}\left(g^{n+1}-g^{n},g^{n}\right)\\
&=\frac{\tau}{\nu}\left(\|g^{n}\|^2-\|g^{n+1}\|^2+\|g^{n+1}-g^{n}\|^2\right)\\
&=\frac{\tau}{\nu}\|g^{n}\|^2-\frac{\tau}{\nu}\|g^{n+1}\|^2+\nu\tau\|\nabla\cdot\tilde{\mathbf{u}}^{n+1}\|^2.
\end{aligned}
\tag{3.51}
$$

Thanks to the identity

$$
\|\nabla\times\mathbf{v}\|^2+\|\nabla\cdot\mathbf{v}\|^2=\|\nabla\mathbf{v}\|^2,\quad\forall\mathbf{v}\in\mathbf{H}_0^1(\Omega), \tag{3.52}
$$

we have

$$
2\tau\left(\tilde{\mathbf{u}}^{n+1},\nabla g^{n}\right)=\frac{\tau}{\nu}\|g^{n}\|^2-\frac{\tau}{\nu}\|g^{n+1}\|^2+\nu\tau\|\nabla\tilde{\mathbf{u}}^{n+1}\|^2-\nu\tau\|\nabla\times\tilde{\mathbf{u}}^{n+1}\|^2. \tag{3.53}
$$

Multiplying (3.41) by 2τ and using (3.44), we get

$$
\begin{aligned}
&2q^{n+1}\left(3q^{n+1}-4q^{n}+q^{n-1}\right)\\
&=\left(3\mathbf{u}^{n+1}-4\mathbf{u}^{n}+\mathbf{u}^{n-1},\mathbf{u}^{n+1}\right)\\
&\quad+\left(\frac{2\tau q^{n+1}}{\sqrt{E_2(\hat{\mathbf{u}}^{n+1})+C_2}}\hat{\mathbf{u}}^{n+1}\cdot\nabla\hat{\mathbf{u}}^{n+1},\tilde{\mathbf{u}}^{n+1}\right).
\end{aligned}
\tag{3.54}
$$

Then, we have

$$
\begin{aligned}
&|q^{n+1}|^2+|2q^{n+1}-q^n|^2-|q^n|^2-|2q^n-q^{n-1}|^2+|q^{n+1}-2q^n+q^{n-1}|^2\\
=&\frac{1}{2}\|\mathbf{u}^{n+1}\|^2+\frac{1}{2}\|2\mathbf{u}^{n+1}-\mathbf{u}^n\|^2-\frac{1}{2}\|\mathbf{u}^n\|^2-\frac{1}{2}\|2\mathbf{u}^n-\mathbf{u}^{n-1}\|^2\\
&+\frac{1}{2}\|\mathbf{u}^{n+1}-2\mathbf{u}^n+\mathbf{u}^{n-1}\|^2+\left(\frac{2\tau q^{n+1}}{\sqrt{E_2(\hat{\mathbf{u}}^{n+1})+C_2}}\hat{\mathbf{u}}^{n+1}\cdot\nabla\hat{\mathbf{u}}^{n+1},\tilde{\mathbf{u}}^{n+1}\right).
\end{aligned}
\tag{3.55}
$$

Next, this is obtained by combining (3.46), (3.47), (3.50), (3.53) with (3.55),

$$
\begin{aligned}
&|q^{n+1}|^2+|2q^{n+1}-q^n|^2-|q^n|^2-|2q^n-q^{n-1}|^2+|q^{n+1}-2q^n+q^{n-1}|^2\\
&+\frac{3}{2}\|\tilde{\mathbf{u}}^{n+1}-\mathbf{u}^{n+1}\|^2+\frac{2}{3}\tau^2\left(\|\nabla H^{n+1}\|^2-\|\nabla H^n\|^2\right)+\frac{\tau}{\nu}\left(\|g^{n+1}\|^2-\|g^n\|^2\right)\\
=&\frac{2\tau r^{n+1}}{\sqrt{E_1\left(\hat{\phi}^{n+1}\right)+C_1}}\left(\hat{\mu}^{n+1}\nabla\hat{\phi}^{n+1},\ \tilde{\mathbf{u}}^{n+1}\right)-\nu\tau\|\nabla\tilde{\mathbf{u}}^{n+1}\|^2-\nu\tau\|\nabla\times\tilde{\mathbf{u}}^{n+1}\|^2.
\end{aligned}
\tag{3.56}
$$

Nextly, combining (3.56) with (3.45), we have

$$
\begin{aligned}
&\frac{\lambda}{2}\left(\|\nabla\phi^{n+1}\|^2+\|2\nabla\phi^{n+1}-\nabla\phi^n\|^2-\|\nabla\phi^n\|^2-\|2\nabla\phi^n-\nabla\phi^{n-1}\|^2\right)\\
&+\frac{\lambda\gamma}{2}\left(\|\phi^{n+1}\|^2+\|2\phi^{n+1}-\phi^n\|^2-\|\phi^n\|^2-\|2\phi^n-\phi^{n-1}\|^2\right)\\
&+\lambda\left(|r^{n+1}|^2+|2r^{n+1}-r^n|^2-|r^n|^2-|2r^n-r^{n-1}|^2\right)\\
&+\frac{\lambda}{2}\left\|\nabla\phi^{n+1}-2\nabla\phi^n+\nabla\phi^{n-1}\right\|^2+\frac{\lambda\gamma}{2}\|\phi^{n+1}-2\phi^n+\phi^{n-1}\|^2\\
&+\lambda|r^{n+1}-2r^n+r^{n-1}|^2+\frac{3}{2}\|\tilde{\mathbf{u}}^{n+1}-\mathbf{u}^{n+1}\|^2+|q^{n+1}|^2\\
&+|2q^{n+1}-q^n|^2-|q^n|^2-|2q^n-q^{n-1}|^2+|q^{n+1}-2q^n+q^{n-1}|^2\\
&+\frac{2}{3}\tau^2\left(\|\nabla H^{n+1}\|^2-\|\nabla H^n\|^2\right)+\frac{\tau}{\nu}\left(\|g^{n+1}\|^2-\|g^n\|^2\right)\\
=&-\nu\tau\|\nabla\times\tilde{\mathbf{u}}^{n+1}\|^2-\nu\tau\|\nabla\tilde{\mathbf{u}}^{n+1}\|^2-2M\tau\|\nabla\mu^{n+1}\|^2.
\end{aligned}
\tag{3.57}
$$

Thus,

$$
\begin{aligned}
&\tilde{E}^{n+1}(\phi,\mathbf{u},r,q)-\tilde{E}^n(\phi,\mathbf{u},r,q)\\
\le&-\nu\tau\|\nabla\times\tilde{\mathbf{u}}^{n+1}\|^2-\nu\tau\|\nabla\tilde{\mathbf{u}}^{n+1}\|^2-2M\tau\|\nabla\mu^{n+1}\|^2.
\end{aligned}
\tag{3.58}
$$

The proof of the theorem is now complete. $\square$

4. Error analysis

The section provides an error analysis for the first-order semi-discrete scheme (3.7)-(3.14). The framework and techniques of error analysis for the SAV approach are well established in [20,23,26,27,33]. Let $(\phi,\mu,\mathbf{u},p,r,q)$ be the exact solution of (3.5), and $(\phi^{n+1}, \mu^{n+1}, \tilde{\mathbf{u}}^{n+1},\mathbf{u}^{n+1}, p^{n+1}, r^{n+1}, q^{n+1})$ be the solution of the scheme (3.7)-(3.14), we denote

$$\begin{aligned} &\tilde{e}_{\mathbf{u}}^{n+1}=\tilde{\mathbf{u}}^{n+1}-\mathbf{u}(t^{n+1}), \quad e_{\mathbf{u}}^{n+1}=\mathbf{u}^{n+1}-\mathbf{u}(t^{n+1}),\\ &e_p^{n+1}=p^{n+1}-p(t^{n+1}), \quad e_q^{n+1}=q^{n+1}-q(t^{n+1}),\\ &e_\phi^{n+1}=\phi^{n+1}-\phi(t^{n+1}), \quad e_\mu^{n+1}=\mu_{n+1}-\mu(t^{n+1}),\\ &e_r^{n+1}=r^{n+1}-r(t^{n+1}). \end{aligned} \tag{4.1}$$

4.1. Error estimate for $\mathbf{u}$, ϕ and μ

The main reslut is stated in Theorem 4.1. We assume that the variables of CHNS model satisfy the following regularity:

$$\begin{aligned} &\phi\in W^{2,\infty}\left(0,T;L^2(\Omega)\right)\cap W^{1,\infty}\left(0,T;H^1(\Omega)\right),\\ &\mu\in W^{1,\infty}\left(0,T;H^1(\Omega)\right)\cap L^{\infty}\left(0,T;H^2(\Omega)\right),\\ &\mathbf{u}\in W^{2,\infty}\left(0,T;\mathbf{H}^{-1}(\Omega)\right)\cap W^{1,\infty}\left(0,T;\mathbf{H}^2(\Omega)\right),\\ &p\in W^{1,\infty}\left(0,T;H^1(\Omega)\right). \end{aligned} \tag{4.2}$$

Theorem 4.1. *Under the assumption of the regularity* (4.2)*, for the first-order scheme* (3.7)-(3.14)*, for all* $0\le n\le N-1$*, we get*

$$\begin{aligned} &\left\|\nabla e_\phi^{m+1}\right\|^2+\left\|e_\phi^{m+1}\right\|^2+\tau\sum_{n=0}^m\left\|\nabla e_\mu^{n+1}\right\|^2+\tau\sum_{n=0}^m\left\|e_\mu^{n+1}\right\|^2+\left|e_r^{m+1}\right|^2\\ &\quad+\left\|e_{\mathbf{u}}^{m+1}\right\|^2+\nu\tau\sum_{n=0}^m\left\|\nabla\tilde{e}_{\mathbf{u}}^{n+1}\right\|^2+\tau\left\|\nabla e_p^{m+1}\right\|^2+\left|e_q^{m+1}\right|^2\\ &\quad+\sum_{n=0}^m\left\|\nabla e_\phi^{n+1}-\nabla e_\phi^n\right\|^2+\sum_{n=0}^m\left\|e_\phi^{n+1}-e_\phi^n\right\|^2+\sum_{n=0}^m\left\|\tilde{e}_{\mathbf{u}}^{n+1}-e_{\mathbf{u}}^n\right\|^2\\ &\quad+\sum_{n=0}^m\left|e_r^{n+1}-e_r^n\right|^2+\sum_{n=0}^m\left|e_q^{n+1}-e_q^n\right|^2\le C\tau^2, \end{aligned} \tag{4.3}$$

where the positive constant C is independent of τ.

We shall follow the steps in the stability proof of Theorem 4.1. The evidence of Theorem 4.1 is similar to [26, Theorem 4.1], with the difference being the error estimate of the velocity $\mathbf{u}$, the scalar auxiliary variable q and the pressure p. Therefore, we need to introduce [22, Lemmas 4.1 and 4.2], which are shown below.

We shall first derive an $H^2(\Omega)$ bound for ϕ^n without assuming the Lipschitz condition on $F(\phi)$. A key ingredient is the following stability result [22]:

$$\begin{aligned} &\|\mathbf{u}^{n+1}\|^2+\nu\tau\sum_{k=0}^n\|\tilde{\mathbf{u}}^{k+1}\|_1^2+\tau\sum_{k=0}^n\|\nabla\mu^{k+1}\|^2\\ &\quad+\|\phi^{n+1}\|_{H^1}^2+|r^{n+1}|^2+|q^{n+1}|^2\le K_1, \end{aligned} \tag{4.4}$$

where the positive constant K_1 is dependent on $\mathbf{u}^0$ and ϕ^0, which can be derived from the unconditionally energy stability (3.19).

Lemma 4.1. *For all* $0 \le n \le N-1$*, there exists a positive constant* K_2 *independent of* τ *such that*

$$\|\Delta\phi^{n+1}\|^2 + \|\mu^{n+1}\|^2 \le K_2. \tag{4.5}$$

Lemma 4.2. *Under the assumption of the regularity* (4.2)*, for all* $0 \le n \le N-1$*, we obtain*

$$\begin{aligned}
&\frac{\lambda}{2\tau}\left(\|\nabla e_\phi^{n+1}\|^2 - \|\nabla e_\phi^n\|^2 + \|\nabla e_\phi^{n+1} - \nabla e_\phi^n\|^2\right) + \frac{M}{4}\|\nabla e_\mu^{n+1}\|^2 \\
&\quad + \frac{\lambda\gamma+1}{2\tau}\left(\|e_\phi^{n+1}\|^2 - \|e_\phi^n\|^2 + \|e_\phi^{n+1} - e_\phi^n\|^2\right) + \frac{M}{4}\|e_\mu^{n+1}\|^2 \\
&\quad + \frac{\lambda}{\tau}\left(|e_r^{n+1}|^2 - |e_r^n|^2 + |e_r^{n+1} - e_r^n|^2\right) \\
&\le C\|e_\phi^n\|^2 + C\|\nabla e_\phi^n\|^2 + C\|\nabla e_\phi^{n+1}\|^2 + C\|e_u^n\|^2 + C\|e_u^{n+1}\|^2 \\
&\quad + C(\tau)^2\|\nabla(e_p^{n+1} - e_p^n)\|^2 + \left(C + \frac{1}{4K_1}\|\nabla\mu^n\|^2\right)|e_r^{n+1}|^2 \\
&\quad + \frac{M}{8}\|e_\mu^n\|^2 + \frac{M}{8}\|\nabla e_\mu^n\|^2 + C\tau^2,
\end{aligned} \tag{4.6}$$

where C *is a positive constant independent of* τ*.*

For details of the proofs of Lemmas 4.1 and 4.2 the reader is referred to [22].

Lemma 4.3. *Under the assumption of the regularity* (4.2)*, for all* $0 \le n \le N-1$*, we obtain*

$$\begin{aligned}
&\frac{\|e_{\mathbf{u}}^{n+1}\|^2 - \|e_{\mathbf{u}}^n\|^2}{2\tau} + \frac{\|\tilde{e}_{\mathbf{u}}^{n+1} - e_{\mathbf{u}}^n\|^2}{2\tau} + \frac{\nu}{2}\|\nabla\tilde{e}_{\mathbf{u}}^{n+1}\|^2 + \frac{\tau}{2}\left(\|\nabla e_p^{n+1}\|^2 - \|\nabla e_p^n\|^2\right) \\
&\le -\left(\frac{2e_q^{n+1}}{\sqrt{E_2(\mathbf{u}^n)+C_2}}\mathbf{u}^n\cdot\nabla\mathbf{u}^n, \tilde{e}_{\mathbf{u}}^{n+1}\right) + C\left(\|\mathbf{u}\|^2_{L^\infty(0,T;H^2(\Omega))} + \|e_{\mathbf{u}}^n\|_1^2\right)\|e_{\mathbf{u}}^n\|^2 \\
&\quad + \frac{M}{16}\|e_\mu^n\|^2 + \frac{M}{16}\|\nabla e_\mu^n\|^2 + C\|\nabla e_\phi^n\|^2 + C\|e_\phi^n\|^2 + C|e_r^{n+1}|^2 + C\|e_{\mathbf{u}}^{n+1}\|^2 \\
&\quad + C\tau^2\left(\|\nabla e_p^n\|^2 + \|\nabla e_p^{n+1}\|^2\right) + C\tau^2,
\end{aligned} \tag{4.7}$$

where the positive constant C *is independent of* τ*.*

Proof. The truncation error $R_{\mathbf{u}}^{n+1}$ is defined as

$$R_{\mathbf{u}}^{n+1} = \frac{\partial\mathbf{u}(t^{n+1})}{\partial t} - \frac{\mathbf{u}(t^{n+1}) - \mathbf{u}(t^n)}{\tau} = \frac{1}{\tau}\int_{t^n}^{t^{n+1}}(t^n - t)\frac{\partial^2\mathbf{u}}{\partial t^2}dt. \tag{4.8}$$

Subtracting (3.5d) at t^{n+1} from (3.10), we get

$$\begin{aligned}
&\frac{\tilde{e}_{\mathbf{u}}^{n+1}-e_{\mathbf{u}}^{n}}{\tau}-\nu\Delta\tilde{e}_{\mathbf{u}}^{n+1}\\
=&\frac{q(t^{n+1})}{\sqrt{E_2(\mathbf{u}(t^{n+1}))+C_2}}\left(\left(\mathbf{u}(t^{n+1})\cdot\nabla\right)\mathbf{u}(t^{n+1})\right)\\
&-\frac{q^{n+1}}{\sqrt{E_2(\mathbf{u}^n)+C_2}}\mathbf{u}^n\cdot\nabla\mathbf{u}^n-\nabla\left(p^n-p(t^{n+1})\right)\\
&+\frac{r^{n+1}}{\sqrt{E_1(\phi^n)+C_1}}\mu^n\nabla\phi^n-\frac{r(t^{n+1})}{\sqrt{E_1(\phi(t^{n+1}))+C_1}}\mu(t^{n+1})\nabla\phi(t^{n+1})+R_{\mathbf{u}}^{n+1}.
\end{aligned}\tag{4.9}$$

Taking the inner product of (4.9) with $\tilde{e}_{\mathbf{u}}^{n+1}$, we get

$$\begin{aligned}
&\frac{\|\tilde{e}_{\mathbf{u}}^{n+1}\|^2-\|e_{\mathbf{u}}^{n}\|^2+\|\tilde{e}_{\mathbf{u}}^{n+1}-e_{\mathbf{u}}^{n}\|^2}{2\tau}+\nu\left\|\nabla\tilde{e}_{\mathbf{u}}^{n+1}\right\|^2\\
=&\left(\frac{q(t^{n+1})}{\sqrt{E_2(\mathbf{u}(t^{n+1}))+C_2}}\left((\mathbf{u}(t^{n+1})\cdot\nabla)\mathbf{u}(t^{n+1})\right)-\frac{q^{n+1}}{\sqrt{E_2(\mathbf{u}^n)+C_2}}\mathbf{u}^n\cdot\nabla\mathbf{u}^n,\tilde{e}_{\mathbf{u}}^{n+1}\right)\\
&-\left(\nabla\left(p^n-p(t^{n+1})\right),\tilde{e}_{\mathbf{u}}^{n+1}\right)\\
&+\left(\frac{r^{n+1}}{\sqrt{E_1(\phi^n)+C_1}}\mu^n\nabla\phi^n-\frac{r(t^{n+1})}{\sqrt{E_1(\phi(t^{n+1}))+C_1}}\mu(t^{n+1})\nabla\phi(t^{n+1}),\tilde{e}_{\mathbf{u}}^{n+1}\right)\\
&+\left(R_{\mathbf{u}}^{n+1},\tilde{e}_{\mathbf{u}}^{n+1}\right).
\end{aligned}\tag{4.10}$$

We observe that (3.13) yields

$$\frac{e_{\mathbf{u}}^{n+1}-\tilde{e}_{\mathbf{u}}^{n+1}}{\tau}+\nabla\left(p^{n+1}-p^n\right)=0,\tag{4.11}$$

and

$$\begin{aligned}
\tilde{e}_{\mathbf{u}}^{n+1}&=e_{\mathbf{u}}^{n+1}+\tau\nabla\left(p^{n+1}-p^n\right)\\
&=e_{\mathbf{u}}^{n+1}+\tau\nabla\left(e_p^{n+1}-e_p^n\right)+\tau\nabla\left(p(t^{n+1})-p(t^n)\right).
\end{aligned}\tag{4.12}$$

Taking the inner product of (4.11) with $(e_{\mathbf{u}}^{n+1}+\tilde{e}_{\mathbf{u}}^{n+1})/2$, we get

$$\frac{\|e_{\mathbf{u}}^{n+1}\|^2-\|\tilde{e}_{\mathbf{u}}^{n+1}\|^2}{2\tau}+\frac{1}{2}\left(\nabla(p^{n+1}-p^n),\tilde{e}_{\mathbf{u}}^{n+1}\right)=0.\tag{4.13}$$

Adding (4.10) and (4.13), we have

$$\begin{aligned}
&\frac{\|e_{\mathbf{u}}^{n+1}\|^2-\|e_{\mathbf{u}}^{n}\|^2+\|\tilde{e}_{\mathbf{u}}^{n+1}-e_{\mathbf{u}}^{n}\|^2}{2\tau}+\nu\|\nabla\tilde{e}_{\mathbf{u}}^{n+1}\|^2\\
=&\left(\frac{q(t^{n+1})}{\sqrt{E_2(\mathbf{u}(t^{n+1}))+C_2}}\left((\mathbf{u}(t^{n+1})\cdot\nabla)\mathbf{u}(t^{n+1})\right)-\frac{q^{n+1}}{\sqrt{E_2(\mathbf{u}^n)+C_2}}\mathbf{u}^n\cdot\nabla\mathbf{u}^n,\tilde{e}_{\mathbf{u}}^{n+1}\right)
\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{2}\left(\nabla(p^{n+1}+p^n-2p(t^{n+1})),\tilde{e}_{\mathbf{u}}^{n+1}\right)\\
&+\left(\frac{r^{n+1}}{\sqrt{E_1(\phi^n)+C_1}}\mu^n\nabla\phi^n-\frac{r(t^{n+1})}{\sqrt{E_1(\phi(t^{n+1}))+C_1}}\mu(t^{n+1})\nabla\phi(t^{n+1}),\tilde{e}_{\mathbf{u}}^{n+1}\right)\\
&+\left(R_{\mathbf{u}}^{n+1},\tilde{e}_{\mathbf{u}}^{n+1}\right).
\end{aligned}\tag{4.14}
$$

Then, we estimate each term of (4.14). For the first item on the right-hand side of (4.14), we obtain

$$
\begin{aligned}
&\left(\frac{q(t^{n+1})}{\sqrt{E_2(\mathbf{u}(t^{n+1}))+C_2}}\left((\mathbf{u}(t^{n+1})\cdot\nabla)\mathbf{u}(t^{n+1})\right)-\frac{q^{n+1}}{\sqrt{E_2(\mathbf{u}^n)+C_2}}\mathbf{u}^n\cdot\nabla\mathbf{u}^n,\tilde{e}_{\mathbf{u}}^{n+1}\right)\\
=&\left(\frac{q(t^{n+1})}{\sqrt{E_2(\mathbf{u}(t^{n+1}))+C_2}}\left((\mathbf{u}(t^{n+1})\cdot\nabla)\mathbf{u}(t^{n+1})\right)-\frac{q(t^{n+1})}{\sqrt{E_2(\mathbf{u}^n)+C_2}}\mathbf{u}^n\cdot\nabla\mathbf{u}^n,\tilde{e}_{\mathbf{u}}^{n+1}\right)\\
&-\left(\frac{e_q^{n+1}}{\sqrt{E_2(\mathbf{u}^n)+C_2}}\mathbf{u}^n\cdot\nabla\mathbf{u}^n,\tilde{e}_{\mathbf{u}}^{n+1}\right)\\
=&\frac{q(t^{n+1})}{\sqrt{E_2(\mathbf{u}(t^{n+1}))+C_2}}\left((\mathbf{u}(t^{n+1})-\mathbf{u}^n)\cdot\nabla\mathbf{u}(t^{n+1}),\tilde{e}_{\mathbf{u}}^{n+1}\right)\\
&+\frac{q(t^{n+1})}{\sqrt{E_2(\mathbf{u}(t^{n+1}))+C_2}}\left(\mathbf{u}^n\cdot\nabla\left(\mathbf{u}(t^{n+1})-\mathbf{u}^n\right),\tilde{e}_{\mathbf{u}}^{n+1}\right)\\
&+\left(\frac{q(t^{n+1})}{\sqrt{E_2(\mathbf{u}(t^{n+1}))+C_2}}-\frac{q(t^{n+1})}{\sqrt{E_2(\mathbf{u}^n)+C_2}}\right)\left(\mathbf{u}^n\cdot\nabla\mathbf{u}^n,\tilde{e}_{\mathbf{u}}^{n+1}\right)\\
&-\left(\frac{e_q^{n+1}}{\sqrt{E_2(\mathbf{u}^n)+C_2}}\mathbf{u}^n\cdot\nabla\mathbf{u}^n,\tilde{e}_{\mathbf{u}}^{n+1}\right).
\end{aligned}\tag{4.15}
$$

From (2.9) and (2.11), the first term on the right-hand side of (4.15) can be derived that

$$
\begin{aligned}
&\frac{q(t^{n+1})}{\sqrt{E_2(\mathbf{u}(t^{n+1}))+C_2}}\left((\mathbf{u}(t^{n+1})-\mathbf{u}^n)\cdot\nabla\mathbf{u}(t^{n+1}),\tilde{e}_{\mathbf{u}}^{n+1}\right)\\
\le\,& c\,(1+c)\,\|\mathbf{u}(t^{n+1})-\mathbf{u}^n\|\|\mathbf{u}(t^{n+1})\|_1\left\|\nabla\tilde{e}_{\mathbf{u}}^{n+1}\right\|\\
\le\,&\frac{\nu}{8}\left\|\nabla\tilde{e}_{\mathbf{u}}^{n+1}\right\|^2+C\|\mathbf{u}\|^2_{L^\infty(0,T;H^2(\Omega))}\left\|e_{\mathbf{u}}^n\right\|^2\\
&\quad+C\tau^2\|\mathbf{u}\|^2_{L^\infty(0,T;H^2(\Omega))}\|\mathbf{u}\|^2_{W^{1,\infty}(0,T;L^2(\Omega))}.
\end{aligned}\tag{4.16}
$$

Using Cauchy-Schwarz inequality and (4.4), the second term on the right-hand side of (4.15) can be estimated by

$$
\frac{q(t^{n+1})}{\sqrt{E_2(\mathbf{u}(t^{n+1}))+C_2}}\left(\mathbf{u}^n\cdot\nabla\left(\mathbf{u}(t^{n+1})-\mathbf{u}^n\right),\tilde{e}_{\mathbf{u}}^{n+1}\right)
$$

$$
\begin{aligned}
&= \frac{q(t^{n+1})}{\sqrt{E_2(\mathbf{u}(t^{n+1}))+C_2}}\left(\mathbf{u}^n\cdot\nabla\left(\mathbf{u}(t^{n+1})-\mathbf{u}(t^{n+1})\right),\tilde{e}_{\mathbf{u}}^{n+1}\right)\\
&\quad -\frac{q(t^{n+1})}{\sqrt{E_2(\mathbf{u}(t^n))+C_2}}\left(e_{\mathbf{u}}^n\cdot\nabla e_{\mathbf{u}}^n-\mathbf{u}(t^n)\cdot\nabla e_{\mathbf{u}}^n,\tilde{e}_{\mathbf{u}}^{n+1}\right)\\
&\le \frac{\nu}{8}\left\|\nabla\tilde{e}_{\mathbf{u}}^{n+1}\right\|^2+C\left(\|\mathbf{u}\|^2_{L^\infty(0,T;H^2(\Omega))}+\|e_{\mathbf{u}}^n\|_1^2\right)\|e_{\mathbf{u}}^n\|^2\\
&\quad +C\tau^2\|\mathbf{u}\|^2_{W^{1,\infty}(0,T;H^2(\Omega))},
\end{aligned}
\tag{4.17}
$$

and

$$
\begin{aligned}
&\left(\frac{q(t^{n+1})}{\sqrt{E_2(\mathbf{u}(t^{n+1}))+C_2}}-\frac{q(t^{n+1})}{\sqrt{E_2(\mathbf{u}^n)+C_2}}\right)\left(\mathbf{u}^n\cdot\nabla\mathbf{u}^n,\tilde{e}_{\mathbf{u}}^{n+1}\right)\\
&\le C\|\mathbf{u}^n\|\|\mathbf{u}^n\|_1\|\nabla\tilde{e}_{\mathbf{u}}^n\|\\
&\le \frac{\nu}{8}\left\|\nabla\tilde{e}_{\mathbf{u}}^{n+1}\right\|^2+C\tau^2\|\mathbf{u}\|^2_{L^\infty(0,T;H^2(\Omega))}\|\mathbf{u}\|^2_{W^{1,\infty}(0,T;L^2(\Omega))}.
\end{aligned}
\tag{4.18}
$$

Then, we estimate the error in the second term on the right-hand side of (4.14), using (4.4), (4.10), (4.12) and Lemma 4.1

$$
\begin{aligned}
&\left(\frac{r^{n+1}}{\sqrt{E_1(\phi^n)+C_1}}\mu^n\nabla\phi^n-\frac{r(t^{n+1})}{\sqrt{E_1(\phi(t^{n+1}))+C_1}}\mu(t^{n+1})\nabla\phi(t^{n+1}),\tilde{e}_{\mathbf{u}}^{n+1}\right)\\
&=\frac{r^{n+1}}{\sqrt{E_1(\phi^n)+C_1}}\left(\mu^n\nabla\phi^n-\mu(t^{n+1})\nabla\phi(t^{n+1}),\tilde{e}_{\mathbf{u}}^{n+1}\right)\\
&\quad +\left(\frac{r^{n+1}}{\sqrt{E_1(\phi^n)+C_1}}-\frac{r(t^{n+1})}{\sqrt{E_1(\phi(t^{n+1}))+C_1}}\right)\left(\mu(t^{n+1})\nabla\phi(t^{n+1}),\tilde{e}_{\mathbf{u}}^{n+1}\right)\\
&\le C\left\|\mu^n-\mu(t^{n+1})\right\|_{L^4}\left\|\nabla\phi^n\right\|_{L^4}\left\|\tilde{e}_{\mathbf{u}}^{n+1}\right\|\\
&\quad +C\left\|\mu(t^{n+1})\right\|_{L^\infty(\Omega)}\left\|\nabla\phi^n-\nabla\phi(t^{n+1})\right\|\left\|\tilde{e}_{\mathbf{u}}^{n+1}\right\|\\
&\quad +\left(\frac{r(t^{n+1})}{\sqrt{E_1(\phi^n)+C_1}}-\frac{r(t^{n+1})}{\sqrt{E_1(\phi(t^{n+1}))+C_1}}\right)\left(\mu(t^{n+1})\nabla\phi(t^{n+1}),\tilde{e}_{\mathbf{u}}^{n+1}\right)\\
&\quad +\frac{e_r^{n+1}}{\sqrt{E_1(\phi^n)+C_1}}\left(\mu(t^{n+1})\nabla\phi(t^{n+1}),\tilde{e}_{\mathbf{u}}^{n+1}\right)\\
&\le \frac{\nu}{8}\left\|\nabla\tilde{e}_{\mathbf{u}}^{n+1}\right\|^2+\frac{M}{16}\left\|e_\mu^n\right\|^2+\frac{M}{16}\left\|\nabla e_\mu^n\right\|^2+C\left\|\nabla e_\phi^n\right\|^2+C\left\|e_\phi^n\right\|^2+C\left|e_r^{n+1}\right|^2\\
&\quad +C\left\|e_{\mathbf{u}}^{n+1}\right\|^2+C\tau^2\left\|\nabla\left(e_p^{n+1}-e_p^n\right)\right\|^2+C\tau^4\|p\|^2_{W^{1,\infty}(0,T;H^1(\Omega))}\\
&\quad +C\tau^2\left(\|\mu\|^2_{W^{1,\infty}(0,T;H^1(\Omega))}+\|\mu\|^2_{L^\infty(0,T;H^2(\Omega))}\right)\\
&\quad +C\tau^2\|\phi\|^2_{W^{1,\infty}(0,T;H^1(\Omega))}.
\end{aligned}
\tag{4.19}
$$

Next, we estimate the third term on the right-hand side of (4.14). Using (4.12), we have

$$
\begin{aligned}
&-\frac{1}{2}\left(\nabla(p^{n+1}+p^n-2p(t^{n+1})),\tilde{e}_{\mathbf{u}}^{n+1}\right)\\
=&-\frac{1}{2}\left(\nabla(e_p^{n+1}+e_p^n-p(t^{n+1})+p(t^n)),\tilde{e}_{\mathbf{u}}^{n+1}\right)\\
=&-\frac{1}{2}\left(\nabla(e_p^{n+1}+e_p^n-p(t^{n+1})+p(t^n)),e_{\mathbf{u}}^{n+1}\right)\\
&+\tau\left(\nabla(e_p^{n+1}-e_p^n)+\nabla(p(t^{n+1})-p(t^n))\right)\\
=&-\frac{\tau}{2}\left(\left\|\nabla e_p^{n+1}\right\|^2-\left\|\nabla e_p^n\right\|^2\right)-\tau\left(\nabla(p(t^{n+1})-p(t^n)),\nabla e_p^n\right)\\
&+\frac{\tau}{2}\left\|\nabla\left(p(t^{n+1})-p(t^n)\right)\right\|^2\\
\leq&-\frac{\tau}{2}\left(\left\|\nabla e_p^{n+1}\right\|^2-\left\|\nabla e_p^n\right\|^2\right)+\tau^2\left\|\nabla e_p^n\right\|^2+C(\tau^2+\tau^3)\|p\|^2_{W^{1,\infty}(0,T;H^1(\Omega))}.
\end{aligned}
\tag{4.20}
$$

For the last term on the right-hand side of (4.14), we obtain

$$
\left(R_{\mathbf{u}}^{n+1},\tilde{e}_{\mathbf{u}}^{n+1}\right)\leq\frac{\nu}{8}\left\|\nabla\tilde{e}_{\mathbf{u}}^{n+1}\right\|^2+C\tau^2\|\mathbf{u}\|^2_{W^{2,\infty}(0,T;H^{-1}(\Omega))}. \tag{4.21}
$$

Combining (4.10) with (4.13)-(4.21), we obtain

$$
\begin{aligned}
&\frac{\|e_{\mathbf{u}}^{n+1}\|^2-\|e_{\mathbf{u}}^n\|^2}{2\tau}+\frac{\|\tilde{e}_{\mathbf{u}}^{n+1}-e_{\mathbf{u}}^n\|^2}{2\tau}+\frac{\nu}{2}\left\|\nabla\tilde{e}_{\mathbf{u}}^{n+1}\right\|^2+\frac{\tau}{2}\left(\left\|\nabla e_p^{n+1}\right\|^2-\left\|\nabla e_p^n\right\|^2\right)\\
\leq&-\frac{e_q^{n+1}}{\sqrt{E_2(\mathbf{u}^n)+C_2}}\left(\left(\mathbf{u}^n\cdot\nabla\right)\mathbf{u}^n,\tilde{e}_{\mathbf{u}}^{n+1}\right)+C\left(\|\mathbf{u}\|^2_{L^\infty(0,T;H^2(\Omega))}+\|e_{\mathbf{u}}^n\|_1^2\right)\|e_{\mathbf{u}}^n\|^2\\
&+\frac{M}{16}\left\|e_\mu^n\right\|^2+\frac{M}{16}\left\|\nabla e_\mu^n\right\|^2+C\left\|\nabla e_\phi^n\right\|^2+C\left\|e_\phi^n\right\|^2\\
&+C\left|e_r^{n+1}\right|^2+C\left\|e_{\mathbf{u}}^{n+1}\right\|^2+C\tau^2\left(\left\|\nabla e_p^n\right\|^2+\left\|\nabla e_p^{n+1}\right\|^2\right)\\
&+C\left(\tau^2+\tau^3+\tau^4\right)\|p\|^2_{W^{1,\infty}(0,T;H^1(\Omega))}+C\tau^2\|\phi\|^2_{W^{1,\infty}(0,T;H^1(\Omega))}\\
&+C\left(\|\mu\|^2_{W^{1,\infty}(0,T;H^1(\Omega))}+(\tau)^2\|\mu\|^2_{L^\infty(0,T;H^2(\Omega))}\right)\\
&+C\left(\|\mathbf{u}\|^2_{W^{2,\infty}(0,T;H^{-1}(\Omega))}+(\tau)^2\|\mathbf{u}\|^2_{W^{1,\infty}(0,T;H^2(\Omega))}\right),
\end{aligned}
\tag{4.22}
$$

which implies the desired result. □

Lemma 4.4. *Under the assumption of the regularity* (4.2)*, for all* $0\leq n\leq N-1$*, we obtain*

$$
\begin{aligned}
&\frac{|e_q^{n+1}|^2-|e_q^n|^2+|e_q^{n+1}-e_q^n|^2}{2\tau}\\
\leq&\left(\frac{e_q^{n+1}}{\sqrt{E_2(\mathbf{u}^n)+C_2}}\mathbf{u}^n\cdot\nabla\mathbf{u}^n,\tilde{e}_{\mathbf{u}}^{n+1}\right)\\
&+C\|\mathbf{u}^n\|_1^2\|e_{\mathbf{u}}^n\|^2+\frac{\nu}{2}\|\nabla\tilde{e}_{\mathbf{u}}^{n+1}\|^2+C\|\nabla\mathbf{u}^n\|^2+C|e_q^{n+1}|^2\\
&+C\tau^2\|\mathbf{u}\|^2_{W^{0,\infty}(0,T;L^2(\Omega))}\|\mathbf{u}\|^2_{L^\infty(0,T;H^2(\Omega))}+C\tau^2|q|^2_{W^{2,\infty}(0,T)},
\end{aligned}
\tag{4.23}
$$

where the positive constant C *is independent of* τ*.*

Proof. Recalling (3.5f) and (3.14), we have

$$\frac{\partial q}{\partial t}=\frac{1}{2q}\mathbf{u}\frac{\partial \mathbf{u}}{\partial t}+\frac{1}{\sqrt{E_2(\mathbf{u})+C_2}}\int_\Omega \mathbf{u}\cdot\nabla\mathbf{u}\cdot\mathbf{u}\,d\mathbf{x}, \tag{4.24}$$

and

$$\frac{q^{n+1}-q^n}{\tau}=\left(\frac{\tilde{\mathbf{u}}^{n+1}-\mathbf{u}^n}{2\tau q^{n+1}}+\frac{1}{\sqrt{E_2(\mathbf{u}^n)+C_2}}\mathbf{u}^n\cdot\nabla\mathbf{u}^n,\tilde{\mathbf{u}}^{n+1}\right). \tag{4.25}$$

Subtracting (4.24) at t^{n+1} from (4.25) leads to

$$\begin{aligned}\frac{e_q^{n+1}-e_q^n}{\tau}=&\left(\frac{1}{\sqrt{E_2(\mathbf{u}^n)+C_2}}\mathbf{u}^n\cdot\nabla\mathbf{u}^n,\tilde{\mathbf{u}}^{n+1}\right)\\&-\frac{1}{\sqrt{E_2(\mathbf{u}(t^{n+1}))}}\left(\mathbf{u}(t^{n+1})\cdot\nabla\mathbf{u}(t^{n+1}),\mathbf{u}(t^{n+1})\right)\\&+\frac{1}{2q^{n+1}}\frac{(\tilde{\mathbf{u}}^{n+1}-\mathbf{u}^n,\tilde{\mathbf{u}}^{n+1})}{\tau}-\frac{1}{2q(t^{n+1})}\frac{\partial\mathbf{u}(t^{n+1})}{\partial t}\mathbf{u}(t^{n+1})+R_q^{n+1},\end{aligned} \tag{4.26}$$

where

$$R_q^{n+1}=\frac{\partial q(t^{n+1})}{\partial t}-\frac{q(t^{n+1})-q(t^n)}{\tau}. \tag{4.27}$$

Multiplying both sides of (4.26) by e_q^{n+1} together, we get the following:

$$\begin{aligned}&\frac{|e_q^{n+1}|^2-|e_q^n|^2+|e_q^{n+1}-e_q^n|^2}{2\tau}\\=&\left(\frac{e_q^{n+1}}{\sqrt{E_2(\mathbf{u}^n)+C_2}}\mathbf{u}^n\cdot\nabla\mathbf{u}^n,\tilde{\mathbf{u}}^{n+1}\right)\\&-\frac{e_q^{n+1}}{\sqrt{E_2(\mathbf{u}(t^{n+1}))}}\left(\mathbf{u}(t^{n+1})\cdot\nabla\mathbf{u}(t^{n+1}),\mathbf{u}(t^{n+1})\right)\\&+\frac{e_q^{n+1}}{2q^{n+1}}\frac{(\tilde{\mathbf{u}}^{n+1}-\mathbf{u}^n,\tilde{\mathbf{u}}^{n+1})}{\tau}-\frac{e_q^{n+1}}{2q(t^{n+1})}\frac{\partial\mathbf{u}(t^{n+1})}{\partial t}\mathbf{u}(t^{n+1})+R_q^{n+1}e_q^{n+1}\\=&\left(\frac{e_q^{n+1}}{\sqrt{E_2(\mathbf{u}^n)+C_2}}\mathbf{u}^n\cdot\nabla\mathbf{u}^n,\tilde{e}_{\mathbf{u}}^{n+1}\right)+\left(\frac{e_q^{n+1}}{\sqrt{E_2(\mathbf{u}^n)+C_2}}\mathbf{u}^n\cdot\nabla\mathbf{u}^n,\mathbf{u}(t^{n+1})\right)\\&-\frac{e_q^{n+1}}{\sqrt{E_2(\mathbf{u}(t^{n+1}))}}\left(\mathbf{u}(t^{n+1})\cdot\nabla\mathbf{u}(t^{n+1}),\mathbf{u}(t^{n+1})\right)\\&+\frac{e_q^{n+1}}{2q^{n+1}}\frac{(\tilde{\mathbf{u}}^{n+1}-\mathbf{u}^n,\tilde{\mathbf{u}}^{n+1})}{\tau}-\frac{e_q^{n+1}}{2q(t^{n+1})}\frac{\partial\mathbf{u}(t^{n+1})}{\partial t}\mathbf{u}(t^{n+1})+R_q^{n+1}e_q^{n+1}.\end{aligned} \tag{4.28}$$

Nextly, we estimate the error in each item of (4.28) separately and obtain the last term on the right-hand side of (4.28)

$$R_q^{n+1}e_q^{n+1}\le C\left(|e_q^{n+1}|^2+\tau^2|q|^2_{W^{2,\infty}(0,T)}\right). \tag{4.29}$$

From the principle of energy dissipation, the second and third terms on the right-hand side of (4.28) can be recast into

$$
\begin{aligned}
&\frac{e_q^{n+1}}{\sqrt{E_2(\mathbf{u}^n)+C_2}}\left(\mathbf{u}^n\cdot\nabla\mathbf{u}^n,\mathbf{u}(t^{n+1})\right)-\frac{e_q^{n+1}}{\sqrt{E_2(\mathbf{u}(t^{n+1}))}}\left(\mathbf{u}(t^{n+1})\cdot\nabla\mathbf{u}(t^{n+1}),\mathbf{u}(t^{n+1})\right)\\
=&\frac{e_q^{n+1}}{\sqrt{E_2(\mathbf{u}^n)+C_2}}\left(\mathbf{u}^n\cdot\nabla\mathbf{u}^n-\mathbf{u}(t^{n+1})\cdot\nabla\mathbf{u}(t^{n+1}),\mathbf{u}(t^{n+1})\right)\\
&+\left(\frac{e_q^{n+1}}{\sqrt{E_2(\mathbf{u}^n)+C_2}}-\frac{e_q^{n+1}}{\sqrt{E_2(\mathbf{u}(t^{n+1}))}}\right)\left(\mathbf{u}(t^{n+1})\cdot\nabla\mathbf{u}(t^{n+1}),\mathbf{u}(t^{n+1})\right)\\
=&-\frac{e_q^{n+1}}{\sqrt{E_2(\mathbf{u}(t^{n+1}))+C_2}}\left(\mathbf{u}^n\cdot\nabla\left(\mathbf{u}(t^{n+1})-\mathbf{u}^n\right),\mathbf{u}(t^{n+1})\right)\\
&-\frac{e_q^{n+1}}{\sqrt{E_2(\mathbf{u}(t^{n+1}))+C_2}}\left((\mathbf{u}(t^{n+1})-\mathbf{u}^n)\cdot\nabla\mathbf{u}(t^{n+1}),\mathbf{u}(t^{n+1})\right)\\
&+\left(\frac{e_q^{n+1}}{\sqrt{E_2(\mathbf{u}^n)+C_2}}-\frac{2e_q^{n+1}}{\sqrt{E_2(\mathbf{u}(t^{n+1}))}}\right)\left(\mathbf{u}(t^{n+1})\cdot\nabla\mathbf{u}(t^{n+1}),\mathbf{u}(t^{n+1})\right)\\
\le& C\left|e_q^{n+1}\right|\cdot\|\mathbf{u}^n\|_1\|\mathbf{u}(t^{n+1})-\mathbf{u}^n\|\|\mathbf{u}(t^{n+1})\|+C\|\mathbf{u}(t^{n+1})-\mathbf{u}^n\|\|\mathbf{u}(t^{n+1})\|_1\|\mathbf{u}(t^{n+1})\|\\
\le& C\left|e_q^{n+1}\right|^2+C\|\mathbf{u}^n\|_1^2\|e_{\mathbf{u}}^n\|^2+C\tau^2\|\mathbf{u}\|^2_{W^{0,\infty}(0,T;L^2(\Omega))}\|\mathbf{u}\|^2_{L^\infty(0,T;H^2(\Omega))}\\
&+C\left|e_q^{n+1}\right|^2+C\|e_{\mathbf{u}}^n\|^2+C\tau^2\|\mathbf{u}\|^2_{W^{0,\infty}(0,T;L^2(\Omega))}\|\mathbf{u}\|^2_{L^\infty(0,T;H^2(\Omega))}.
\end{aligned}\tag{4.30}
$$

Then, we estimate the error in the fourth term and observe from (3.13) that $\mathbf{u}^{n+1}=P_{\mathbf{H}}\tilde{\mathbf{u}}^{n+1}$. Using the Cauchy-Scwarz inequality and (2.12) implies that $\|\mathbf{u}^{n+1}\|_1\le C(\Omega)\|\tilde{\mathbf{u}}^{n+1}\|_1$, we obtain

$$
\begin{aligned}
&\frac{e_q^{n+1}}{2q^{n+1}}\frac{(\tilde{\mathbf{u}}^{n+1}-\mathbf{u}^n,\tilde{\mathbf{u}}^{n+1})}{\tau}-\frac{e_q^{n+1}}{2q(t^{n+1})}\frac{\partial\mathbf{u}(t^{n+1})}{\partial t}\mathbf{u}(t^{n+1})\\
=&\frac{e_q^{n+1}}{2q^{n+1}}\left(\frac{\tilde{e}_{\mathbf{u}}^{n+1}+\mathbf{u}(t^{n+1})-\mathbf{u}^n}{\tau},\tilde{\mathbf{u}}^{n+1}\right)-\frac{e_q^{n+1}}{2q(t^{n+1})}\frac{\partial\mathbf{u}(t^{n+1})}{\partial t}\mathbf{u}(t^{n+1})\\
\le& C\left|e_q^{n+1}\right|\|\mathbf{u}^{n+1}\|_1\left(\|\nabla\tilde{e}_{\mathbf{u}}^{n+1}\|+\|\mathbf{u}(t^{n+1})-\mathbf{u}^n\|\right)+C\left|e_q^{n+1}\right|\|\mathbf{u}(t^{n+1})\|\|\nabla\mathbf{u}^n\|\\
\le& C\left|e_q^{n+1}\right|^2+\frac{\nu}{2}\left\|\nabla\tilde{e}_{\mathbf{u}}^{n+1}\right\|^2+C\|\nabla\mathbf{u}^n\|^2+C\|\mathbf{u}^n\|_1^2\|e_{\mathbf{u}}^n\|^2\\
&+C\tau^2\|\mathbf{u}\|^2_{W^{0,\infty}(0,T;L^2(\Omega))}\|\mathbf{u}\|^2_{L^\infty(0,T;H^2(\Omega))}.
\end{aligned}\tag{4.31}
$$

Combining (4.31) and (4.28)-(4.30) yields

$$
\begin{aligned}
&\frac{|e_q^{n+1}|^2-|e_q^n|^2+|e_q^{n+1}-e_q^n|^2}{2\tau}\\
\le&\left(\frac{e_q^{n+1}}{\sqrt{E_2(\mathbf{u}^n)+C_2}}\mathbf{u}^n\cdot\nabla\mathbf{u}^n,\tilde{e}_{\mathbf{u}}^{n+1}\right)+C\|\mathbf{u}^n\|_1^2\|e_{\mathbf{u}}^n\|^2+\frac{\nu}{2}\|\nabla\tilde{e}_{\mathbf{u}}^{n+1}\|^2+C\left\|\nabla\mathbf{u}^n\right\|^2\\
&+C\left|e_q^{n+1}\right|^2+C\tau^2\|\mathbf{u}\|^2_{W^{0,\infty}(0,T;L^2(\Omega))}\|\mathbf{u}\|^2_{L^\infty(0,T;H^2(\Omega))}+C\tau^2|q|^2_{W^{2,\infty}(0,T)}.
\end{aligned}\tag{4.32}
$$

The proof is complete. □

Now, we prove Theorem 4.1 in conjunction with the above lemmas.

Proof of Theorem 4.1. Using Lemmas 4.2-4.4 leads to

$$
\begin{aligned}
&\frac{\lambda}{2\tau}\left(\|\nabla e_\phi^{n+1}\|^2-\|\nabla e_\phi^n\|^2+\|\nabla e_\phi^{n+1}-\nabla e_\phi^n\|^2\right)+\frac{M}{4}\left\|\nabla e_\mu^{n+1}\right\|^2\\
&\quad+\frac{\lambda\gamma+1}{2\tau}\left(\|e_\phi^{n+1}\|^2-\|e_\phi^n\|^2+\|e_\phi^{n+1}-e_\phi^n\|^2\right)+\frac{M}{4}\left\|\nabla e_\mu^{n+1}\right\|^2\\
&\quad+\frac{\lambda}{\tau}\left(|e_r^{n+1}|^2-|e_r^n|^2+|e_r^{n+1}-e_r^n|^2\right)+\frac{\tau}{2}\left(\|\nabla e_p^{n+1}\|^2-\|\nabla e_p^n\|^2\right)\\
&\quad+\frac{\|e_{\mathbf{u}}^{n+1}\|^2-\|e_{\mathbf{u}}^n\|^2+\|\tilde{e}_{\mathbf{u}}^{n+1}-e_{\mathbf{u}}^n\|^2}{2\tau}+\frac{\nu}{2}\left\|\nabla\tilde{e}_{\mathbf{u}}^{n+1}\right\|^2\\
&\quad+\frac{|e_q^{n+1}|^2-|e_q^n|^2+|e_q^{n+1}-e_q^n|^2}{2\tau}\\
\le&\left(C+\frac{1}{4K_1}\|\nabla\mu^n\|^2\right)\left|e_r^{n+1}\right|^2+C\left(\|e_{\mathbf{u}}^n\|_1^2+\|\mathbf{u}^n\|_1^2\right)\|e_{\mathbf{u}}^n\|^2\\
&\quad+C\left\|\nabla e_\phi^{n+1}\right\|^2+C\left\|\nabla e_\phi^n\right\|^2+C\left\|e_\phi^n\right\|^2+C\left\|e_{\mathbf{u}}^{n+1}\right\|^2\\
&\quad+\frac{3}{16}M\left\|e_\mu^n\right\|^2+\frac{3}{16}M\left\|\nabla e_\mu^n\right\|^2+C\left|e_q^{n+1}\right|^2\\
&\quad+C\tau^2\left(\|\nabla e_p^n\|^2+\|\nabla e_p^{n+1}\|^2\right)+C\tau^2. \qquad (4.33)
\end{aligned}
$$

Multiplying (4.33) by 2τ and summing over n, $n=0,1,2,\ldots,m\le N-1$, and applying Lemma 2.1, we can obtain

$$
\begin{aligned}
&\left\|\nabla e_\phi^{m+1}\right\|^2+\left\|e_\phi^{m+1}\right\|^2+\tau\sum_{n=0}^m\left\|\nabla e_\mu^{n+1}\right\|^2+\tau\sum_{n=0}^m\left\|e_\mu^{n+1}\right\|^2+\left|e_r^{n+1}\right|^2\\
&\quad+\left\|e_{\mathbf{u}}^{m+1}\right\|^2+\nu\tau\sum_{n=0}^m\left\|\nabla\tilde{e}_{\mathbf{u}}^{n+1}\right\|^2+\left|e_q^{m+1}\right|^2+\tau\left\|\nabla e_p^{m+1}\right\|^2\\
&\quad+\sum_{n=0}^m\left\|\nabla e_\phi^{n+1}-\nabla e_\phi^n\right\|^2+\sum_{n=0}^m\left\|e_\phi^{n+1}-e_\phi^n\right\|^2+\sum_{n=0}^m\left\|\nabla\tilde{e}_{\mathbf{u}}^{n+1}-\nabla e_{\mathbf{u}}^n\right\|^2\\
&\quad+\sum_{n=0}^m\left\|\nabla e_r^{n+1}-\nabla e_r^n\right\|^2+\sum_{n=0}^m\left\|e_q^{n+1}-e_q^n\right\|^2\le C\tau^2, \qquad (4.34)
\end{aligned}
$$

where we use the fact that

$$
\frac{2\tau}{4K_1}\sum_{n=0}^m\|\nabla\mu^n\|^2\left|e_r^{n+1}\right|^2\le\frac{\tau}{2K_1}\left|e_r^{m+1}\right|^2\sum_{n=0}^m\|\nabla\mu^n\|^2\le\frac{1}{2}\left|e_r^{m+1}\right|^2. \qquad (4.35)
$$

Since

$$
\left|e_r^{m+1}\right|=\max_{0\le m\le N-1}\left|e_r^{m+1}\right| \quad \text{for all} \quad 0\le m\le N-1,
$$

the Eq. (4.34) also implies

$$
\left\|\nabla e_\phi^{m+1}\right\|^2+\left\|e_\phi^{m+1}\right\|^2+\tau\sum_{n=0}^m\left\|\nabla e_\mu^{n+1}\right\|^2+\tau\sum_{n=0}^m\left\|e_\mu^{n+1}\right\|^2+\left|e_r^{n+1}\right|^2
$$

$$
\begin{aligned}
&+\left\|e_{\mathbf{u}}^{m+1}\right\|^2+\nu\tau\sum_{n=0}^{m}\left\|\nabla\tilde{e}_{\mathbf{u}}^{n+1}\right\|^2+\left|e_q^{m+1}\right|^2+\tau\left\|\nabla e_p^{m+1}\right\|^2\\
&+\sum_{n=0}^{m}\left\|\nabla e_\phi^{n+1}-\nabla e_\phi^{n}\right\|^2+\sum_{n=0}^{m}\left\|e_\phi^{n+1}-e_\phi^{n}\right\|^2+\sum_{n=0}^{m}\left\|\nabla\tilde{e}_{\mathbf{u}}^{n+1}-\nabla e_{\mathbf{u}}^{n}\right\|^2\\
&+\sum_{n=0}^{m}\left\|\nabla e_r^{n+1}-\nabla e_r^{n}\right\|^2+\sum_{n=0}^{m}\left\|e_q^{n+1}-e_q^{n}\right\|^2\le C\tau^2,
\end{aligned}\tag{4.36}
$$

which implies the desired result in Theorem 4.1. □

4.2. Error estimate for p

Theorem 4.1 cannot lead to an error estimate on the pressure p. As with the error analysis of the projective scheme, the pressure error must be obtained through the inf-sup condition

$$
\|p\|_{L^2(\Omega)/\mathbb{R}}\le\sup_{\mathbf{v}\in\mathbf{H}_0^1(\Omega)}\frac{(p,\nabla\cdot\mathbf{v})}{\|\mathbf{v}\|_{H^1}}=\sup_{\mathbf{v}\in\mathbf{H}_0^1(\Omega)}\frac{-(\nabla p,\mathbf{v})}{\|\mathbf{v}\|_{H^1}}.\tag{4.37}
$$

This is clearly true in the spatially continuous case. Therefore, we need to estimate $(\nabla e_p^{n+1},\mathbf{v})$, which requires additional estimates. We assume that the variables satisfy the following regularity:

$$
\begin{aligned}
&\phi\in W^{3,\infty}\left(0,T;L^2(\Omega)\right)\cap W^{2,\infty}\left(0,T;H^2(\Omega)\right),\\
&\mu\in W^{1,\infty}\left(0,T;H^1(\Omega)\right)\cap L^{\infty}\left(0,T;H^2(\Omega)\right),\\
&\boldsymbol{u}\in W^{3,\infty}\left(0,T;\boldsymbol{L}^2(\Omega)\right)\cap W^{2,\infty}\left(0,T;\boldsymbol{H}^1(\Omega)\right)\cap L^{\infty}\left(0,T;\boldsymbol{H}^2(\Omega)\right),\\
&p\in W^{2,\infty}\left(0,T;H^1(\Omega)\right).
\end{aligned}\tag{4.38}
$$

Theorem 4.2. *Under the assumption of the regularity* (4.38)*, then for the first-order scheme* (3.7)-(3.14)*, we can obtain*

$$
\tau\sum_{n=0}^{m}\left\|e_p^{n+1}\right\|_{L^2(\Omega)/\mathbb{R}}^2\le C\tau^2,\quad\forall\,0\le m\le N-1,\tag{4.39}
$$

where C is a positive constant independent of τ.

The proof of Theorem 4.2 needs to be accomplished by a series of intermediate lemmas.

Lemma 4.5. *Under the assumption of the regularity* (4.38)*, for all $0\le m\le N-1$, we obtain*

$$
\tau\sum_{n=0}^{m}\left|\delta_\tau e_q^{n+1}\right|^2\le C\tau^2.\tag{4.40}
$$

Proof. Multiplying both sides of (4.26) with $\delta_\tau e_q^{n+1}$ leads to

$$\begin{aligned}\left|\delta_\tau e_q^{n+1}\right|^2 =& \frac{\delta_\tau e_q^{n+1}}{\sqrt{E_2(\mathbf{u}^n)+C_2}}\left(\mathbf{u}^n\cdot\nabla\mathbf{u}^n,\tilde{\mathbf{u}}^{n+1}\right)\\ &-\frac{\delta_\tau e_q^{n+1}}{\sqrt{E_2(\mathbf{u}(t^{n+1}))+C_2}}\left(\mathbf{u}(t^{n+1})\cdot\nabla\mathbf{u}(t^{n+1}),\mathbf{u}(t^{n+1})\right)\\ &+\frac{\delta_\tau e_q^{n+1}}{2q^{n+1}}\left(\frac{\tilde{\mathbf{u}}^{n+1}-\mathbf{u}^n}{\tau},\tilde{\mathbf{u}}^{n+1}\right)-\frac{\delta_\tau e_q^{n+1}}{2q(t^{n+1})}\frac{\partial\mathbf{u}(t^{n+1})}{\partial t}\mathbf{u}(t^{n+1})\\ &+R_q^{n+1}\delta_\tau e_q^{n+1},\end{aligned} \tag{4.41}$$

and we derive that

$$\begin{aligned}\left|\delta_\tau e_q^{n+1}\right|^2 =& \frac{\delta_\tau e_q^{n+1}}{\sqrt{E_2(\mathbf{u}^n)+C_2}}\left(\mathbf{u}^n\cdot\nabla\mathbf{u}^n,\tilde{e}_{\mathbf{u}}^{n+1}\right)\\ &+\frac{\delta_\tau e_q^{n+1}}{\sqrt{E_2(\mathbf{u}^n)+C_2}}\left(\mathbf{u}^n\cdot\nabla\mathbf{u}^n,\mathbf{u}(t^{n+1})\right)\\ &+\frac{\delta_\tau e_q^{n+1}}{2q^{n+1}}\left(\frac{\tilde{\mathbf{u}}^{n+1}-\mathbf{u}^n}{\tau},\tilde{\mathbf{u}}^{n+1}\right)\\ &-\frac{\delta_\tau e_q^{n+1}}{2q(t^{n+1})}\frac{\partial\mathbf{u}(t^{n+1})}{\partial t}\mathbf{u}(t^{n+1})+R_q^{n+1}\delta_\tau e_q^{n+1},\end{aligned} \tag{4.42}$$

and then

$$\begin{aligned}\left|\delta_\tau e_q^{n+1}\right|^2 =& \frac{\delta_\tau e_q^{n+1}}{\sqrt{E_2(\mathbf{u}^n)+C_2}}\left(\mathbf{u}^n\cdot\nabla\mathbf{u}^n,\tilde{e}_{\mathbf{u}}^{n+1}\right)\\ &+\frac{\delta_\tau e_q^{n+1}}{\sqrt{E_2(\mathbf{u}^n)+C_2}}\left(\mathbf{u}^n\cdot\nabla\mathbf{u}^n-\mathbf{u}(t^{n+1})\cdot\nabla\mathbf{u}(t^{n+1}),\mathbf{u}(t^{n+1})\right)\\ &+\frac{\delta_\tau e_q^{n+1}}{2q^{n+1}}\left(\frac{\tilde{\mathbf{u}}^{n+1}-\mathbf{u}^n}{\tau},\tilde{\mathbf{u}}^{n+1}\right)-\frac{\delta_\tau e_q^{n+1}}{2q(t^{n+1})}\frac{\partial\mathbf{u}(t^{n+1})}{\partial t}\mathbf{u}(t^{n+1})\\ &+R_q^{n+1}\delta_\tau e_q^{n+1}.\end{aligned} \tag{4.43}$$

From Theorem 4.1, we obtain

$$\tau\sum_{n=0}^{m}\left\|\nabla\tilde{e}_{\mathbf{u}}^{n+1}\right\|^2\le C\tau^2, \tag{4.44}$$

which implies that

$$\|\mathbf{u}^{n+1}\|_1\le C\|\tilde{\mathbf{u}}^{n+1}\|_1\le C\tau^{\frac{1}{2}}\le K_2. \tag{4.45}$$

Then, the first term on the right-hand side of (4.43) can be bounded by

$$\frac{\delta_\tau e_q^{n+1}}{\sqrt{E_2(\mathbf{u}^n)+C_2}}\left(\mathbf{u}^n\cdot\nabla\mathbf{u}^n,\tilde{e}_{\mathbf{u}}^{n+1}\right)$$

$$\leq C\big|\delta_\tau e_q^{n+1}\big|\|\mathbf{u}^n\|\|\mathbf{u}^n\|_1\big\|\nabla\tilde{e}_{\mathbf{u}}^{n+1}\big\|$$
$$\leq \frac{1}{8}\big|\delta_\tau e_q^{n+1}\big|^2 + C\big\|\nabla\tilde{e}_{\mathbf{u}}^{n+1}\big\|^2. \tag{4.46}$$

The second term on the right-hand side of (4.43) can be estimated by

$$\begin{aligned}
&\frac{\delta_\tau e_q^{n+1}}{\sqrt{E_2(\mathbf{u}^n)+C_2}}\left(\mathbf{u}^n\cdot\nabla\mathbf{u}^n - \mathbf{u}(t^{n+1})\cdot\nabla\mathbf{u}(t^{n+1}),\mathbf{u}(t^{n+1})\right)\\
=&-\frac{\delta_\tau e_q^{n+1}}{\sqrt{E_2(\mathbf{u}^n)+C_2}}\left(\mathbf{u}^n\cdot\nabla\left(\mathbf{u}(t^{n+1})-\mathbf{u}^n\right),\mathbf{u}(t^{n+1})\right)\\
&-\frac{\delta_\tau e_q^{n+1}}{\sqrt{E_2(\mathbf{u}^n)+C_2}}\left(\mathbf{u}(t^{n+1})\cdot\nabla\left(\mathbf{u}(t^{n+1})-\mathbf{u}^n\right),\mathbf{u}(t^{n+1})\right)\\
\leq&\frac{1}{8}\big|\delta_\tau e_q^{n+1}\big|^2 + C\big\|e_{\mathbf{u}}^{n+1}\big\|^2 + C\tau^2.
\end{aligned}\tag{4.47}$$

The third term on the right-hand side of (4.43) can be estimated by

$$\begin{aligned}
&\frac{\delta_\tau e_q^{n+1}}{2q^{n+1}}\left(\frac{\tilde{\mathbf{u}}^{n+1}-\mathbf{u}^n}{\tau},\tilde{\mathbf{u}}^{n+1}\right) - \frac{\delta_\tau e_q^{n+1}}{2q(t^{n+1})}\frac{\partial\mathbf{u}(t^{n+1})}{\partial t}\mathbf{u}(t^{n+1})\\
=&\frac{\delta_\tau e_q^{n+1}}{2q^{n+1}}\left(\frac{\tilde{\mathbf{u}}^{n+1}-\mathbf{u}^n}{\tau},\tilde{e}_{\mathbf{u}}^{n+1}\right)\\
&+\frac{\delta_\tau e_q^{n+1}}{2q^{n+1}}\left(\frac{\tilde{\mathbf{u}}^{n+1}-\mathbf{u}^n}{\tau}-\frac{\partial\mathbf{u}(t^{n+1})}{\partial t},\mathbf{u}(t^{n+1})\right)\\
&+\left(\frac{\delta_\tau e_q^{n+1}}{2q^{n+1}}-\frac{\delta_\tau e_q^{n+1}}{2q(t^{n+1})}\right)\left(\frac{\partial\mathbf{u}(t^{n+1})}{\partial t},\mathbf{u}(t^{n+1})\right)\\
\leq&\frac{1}{8}\big|\delta_\tau e_q^{n+1}\big|^2 + C\big\|\nabla\tilde{e}_{\mathbf{u}}^{n+1}\big\|^2 + C\|\tilde{\mathbf{u}}^{n+1}\|_1^2 + C\tau^2.
\end{aligned}\tag{4.48}$$

The fourth term on the right-hand side of (4.43) can be estimated by

$$R_q^{n+1}\delta_\tau e_q^{n+1} \leq \frac{1}{8}\big|\delta_\tau e_q^{n+1}\big|^2 + C\tau^2\|q\|^2_{W^{2,\infty}(0,T;H^2(\Omega))}. \tag{4.49}$$

Combining (4.46)-(4.49) with (4.43) results in

$$\big|\delta_\tau e_q^{n+1}\big|^2 \leq \frac{1}{2}\big|\delta_\tau e_q^{n+1}\big|^2 + C\big\|\nabla\tilde{e}_{\mathbf{u}}^{n+1}\big\|^2 + C\big\|e_{\mathbf{u}}^n\big\|^2 + C\tau^2\|q\|^2_{W^{2,\infty}(0,T;H^2(\Omega))}. \tag{4.50}$$

Multiplying (4.50) by τ and summing up for n from 0 to m, and recalling Theorem 4.1, we obtain

$$\tau\sum_{n=0}^m\big|\delta_\tau e_q^{n+1}\big|^2 \leq C\tau\sum_{n=0}^m\big\|\nabla\tilde{e}_{\mathbf{u}}^{n+1}\big\|^2 + C\tau\sum_{n=0}^m\big\|e_{\mathbf{u}}^n\big\|^2 + C\tau^2 \leq C\tau^2. \tag{4.51}$$

The proof is complete. □

Lemma 4.6. *Under the assumption of the regularity* (4.38)*, for all* $0 \le m \le N-1$*, we obtain*

$$\begin{aligned}&\left\|\delta_\tau e_\phi^{m+1}\right\|^2 + \sum_{n=1}^{m}\left\|\delta_\tau e_\phi^{n+1} - \delta_\tau e_\phi^{n}\right\|^2 + \tau\sum_{n=1}^{m}\left\|\delta_\tau e_\mu^{n+1}\right\|^2 \\ &\quad + \tau\sum_{n=1}^{m}\left\|\delta_\tau e_r^{n+1}\right\|^2 + \tau\sum_{n=1}^{m}\left\|\delta_\tau \Delta e_\phi^{n+1}\right\|^2 + \tau\left\|\delta_\tau \nabla e_\phi^{m+1}\right\|^2 \\ &\le C\tau\sum_{n=1}^{m}\|\delta_\tau e_{\mathbf{u}}^{n}\|^2 + C\tau^2. \end{aligned} \tag{4.52}$$

The proof of this result is quite similar to that given in [22, Appendix A] and is not complicated but is too long to give here, and so is omitted.

Lemma 4.7. *Under the assumption of the regularity* (4.38)*, for all* $0 \le m \le N-1$*, we obtain*

$$\left\|\delta_\tau e_{\mathbf{u}}^{m+1}\right\|^2 + \nu\tau\sum_{n=1}^{m}\left\|\nabla\delta_\tau \tilde{e}_{\mathbf{u}}^{n+1}\right\|^2 + \tau^2\left\|\nabla\delta_\tau e_p^{m+1}\right\|^2 \le C\tau^2. \tag{4.53}$$

Proof. We now establish an estimate on $\|\delta_\tau e_{\mathbf{u}}^{n+1}\|$. Adding (4.9) and (4.11) results in

$$\begin{aligned}&\frac{e_{\mathbf{u}}^{n+1} - e_{\mathbf{u}}^{n}}{\tau} - \nu\Delta\tilde{e}_{\mathbf{u}}^{n+1} \\ =\ & \frac{q(t^{n+1})}{\sqrt{E_2(\mathbf{u}(t^{n+1})) + C_2}}\left((\mathbf{u}(t^{n+1})\cdot\nabla)\mathbf{u}(t^{n+1})\right) \\ &- \frac{q^{n+1}}{\sqrt{E_2(\mathbf{u}^n) + C_2}}\mathbf{u}^n\cdot\nabla\mathbf{u}^n - \nabla e_p^{n+1} + R_{\mathbf{u}}^{n+1} \\ &+ \frac{r^{n+1}}{\sqrt{E_1(\phi^n) + C_1}}\mu^n\nabla\phi^n - \frac{r(t^{n+1})}{\sqrt{E_1(\phi(t^{n+1})) + C_1}}\mu(t^{n+1})\nabla\phi(t^{n+1}). \end{aligned} \tag{4.54}$$

For $n \ge 1$, then the $n+1$ and n terms of (4.54) are differenced to obtain

$$\begin{aligned}&\delta_{\tau\tau} e_{\mathbf{u}}^{n+1} - \nu\Delta\delta_\tau\tilde{e}_{\mathbf{u}}^{n+1} \\ =\ & \delta_\tau R_{\mathbf{u}}^{n+1} - \nabla\delta_\tau e_p^{n+1} \\ &+ \delta_\tau\left(\frac{q(t^{n+1})}{\sqrt{E_2(\mathbf{u}(t^{n+1})) + C_2}}\left((\mathbf{u}(t^{n+1})\cdot\nabla)\mathbf{u}(t^{n+1})\right) - \frac{q^{n+1}}{\sqrt{E_2(\mathbf{u}^n) + C_2}}\mathbf{u}^n\cdot\nabla\mathbf{u}^n\right) \\ &+ \delta_\tau\left(\frac{r^{n+1}}{\sqrt{E_1(\phi^n) + C_1}}\mu^n\nabla\phi^n - \frac{r(t^{n+1})}{\sqrt{E_1(\phi(t^{n+1})) + C_1}}\mu(t^{n+1})\nabla\phi(t^{n+1})\right). \end{aligned} \tag{4.55}$$

Taking the inner product of (4.55) with $\delta_\tau\tilde{e}_{\mathbf{u}}^{n+1}$, we obtain

$$\begin{aligned}&\left(\delta_{\tau\tau} e_{\mathbf{u}}^{n+1}, \delta_\tau\tilde{e}_{\mathbf{u}}^{n+1}\right) + \nu\left\|\nabla\delta_\tau\tilde{e}_{\mathbf{u}}^{n+1}\right\|^2 \\ =\ & \left(d_t R_{\mathbf{u}}^{n+1}, \delta_\tau\tilde{e}_{\mathbf{u}}^{n+1}\right) - \left(\nabla d_t e_p^{n+1}, \delta_\tau\tilde{e}_{\mathbf{u}}^{n+1}\right)\end{aligned}$$

$$
\begin{aligned}
&+\left(\delta_\tau\left(\frac{q(t^{n+1})}{\sqrt{E_2(\mathbf{u}(t^{n+1}))+C_2}}\left((\mathbf{u}(t^{n+1})\cdot\nabla)\mathbf{u}(t^{n+1})\right)\right.\right.\\
&\qquad\left.\left.-\frac{q^{n+1}}{\sqrt{E_2(\mathbf{u}^n)+C_2}}\mathbf{u}^n\cdot\nabla\mathbf{u}^n\right),\delta_\tau\tilde{e}_{\mathbf{u}}^{n+1}\right)\\
&+\left(\delta_\tau\left(\frac{r^{n+1}}{\sqrt{E_1(\phi^n)+C_1}}\mu^n\nabla\phi^n\right.\right.\\
&\qquad\left.\left.-\frac{r(t^{n+1})}{\sqrt{E_1(\phi(t^{n+1}))+C_1}}\mu(t^{n+1})\nabla\phi(t^{n+1})\right),\delta_\tau\tilde{e}_{\mathbf{u}}^{n+1}\right).
\end{aligned}\tag{4.56}
$$

The first term on the left-hand side of (4.56) can be rewritten as

$$
\left(\delta_{\tau\tau}e_{\mathbf{u}}^{n+1},\delta_\tau\tilde{e}_{\mathbf{u}}^{n+1}\right)=\frac{\|\delta_\tau e_{\mathbf{u}}^{n+1}\|^2-\|\delta_\tau e_{\mathbf{u}}^{n}\|^2+\|\delta_\tau e_{\mathbf{u}}^{n+1}-\delta_\tau e_{\mathbf{u}}^{n}\|^2}{2\tau}.\tag{4.57}
$$

Now, the first term on the right-hand side of (4.56) can be bounded by

$$
\left(\delta_\tau R_{\mathbf{u}}^{n+1},\delta_\tau\tilde{e}_{\mathbf{u}}^{n+1}\right)\le\frac{\nu}{6}\left\|\nabla\delta_\tau\tilde{e}_{\mathbf{u}}^{n+1}\right\|^2+C\|\mathbf{u}\|^2_{W^{3,\infty}(0,T;L^2(\Omega))}.\tag{4.58}
$$

For the second term on the right-hand side of (4.56) can be recast into

$$
-\left(\nabla\delta_\tau e_p^{n+1},\delta_\tau\tilde{e}_{\mathbf{u}}^{n+1}\right)=-\left(\nabla\delta_\tau e_p^{n},\delta_\tau\tilde{e}_{\mathbf{u}}^{n+1}\right)-\left(\nabla(\delta_\tau e_p^{n+1}-\delta_\tau e_p^{n}),\delta_\tau\tilde{e}_{\mathbf{u}}^{n+1}\right).\tag{4.59}
$$

Since we can derive from (4.11) that

$$
\delta_\tau\tilde{e}_{\mathbf{u}}^{n+1}=\delta_\tau e_{\mathbf{u}}^{n+1}+\nabla\left(p^{n+1}-2p^n+p^{n-1}\right).\tag{4.60}
$$

The first term on the right-hand side of (4.59) can be estimated by

$$
\begin{aligned}
-\left(\nabla\delta_\tau e_p^{n},\delta_\tau\tilde{e}_{\mathbf{u}}^{n+1}\right)&=-\left(\nabla\delta_\tau e_p^{n},\nabla(p^{n+1}-2p^n+p^{n-1})\right)\\
&=-\tau\left(\nabla\delta_\tau e_p^{n},\nabla(\delta_\tau e_p^{n+1}-\delta_\tau e_p^{n})\right)\\
&\quad-\left(\nabla\delta_\tau e_p^{n},\nabla\left(p(t^{n+1})-2p(t^n)+p(t^{n-1})\right)\right)\\
&\le-\frac{\tau}{2}\left(\|\nabla\delta_\tau e_p^{n+1}\|^2-\|\nabla\delta_\tau e_p^{n}\|^2-\|\nabla\delta_\tau e_p^{n+1}-\nabla\delta_\tau e_p^{n}\|^2\right)\\
&\quad+\tau^2\|\nabla\delta_\tau e_p^{n}\|^2+C\tau^2\|p\|^2_{W^{2,\infty}(0,T;H^1(\Omega))}.
\end{aligned}\tag{4.61}
$$

The second term on the right-hand side of (4.59) can be transformed by

$$
\begin{aligned}
&-\left(\nabla(\delta_\tau e_p^{n+1}-d_t e_p^{n}),\delta_\tau\tilde{e}_{\mathbf{u}}^{n+1}\right)\\
=&-\left(\nabla(\delta_\tau e_p^{n+1}-\delta_\tau e_p^{n}),\delta_\tau e_{\mathbf{u}}^{n+1}+\nabla(p^{n+1}-2p^n+p^{n-1})\right)\\
=&-\left(\nabla(\delta_\tau e_p^{n+1}-\delta_\tau e_p^{n}),\nabla(p^{n+1}-2p^n+p^{n-1})\right)\\
=&-\left(\nabla(\delta_\tau e_p^{n+1}-\delta_\tau e_p^{n}),\nabla(p(t^{n+1})-2p(t^n)+p(t^{n-1}))\right)\\
&-\tau\left(\nabla(\delta_\tau e_p^{n+1}-\delta_\tau e_p^{n}),\nabla(\delta_\tau e_p^{n+1}-\delta_\tau e_p^{n})\right)\\
\le&-\frac{\tau}{2}\left\|\nabla\delta_\tau e_p^{n+1}-\nabla\delta_\tau e_p^{n}\right\|^2+C\tau^2\|p\|^2_{W^{2,\infty}(0,T;H^1(\Omega))}.
\end{aligned}\tag{4.62}
$$

The third term on the right-hand side of (4.56) can be bounded by using the similar procedure as in Theorem 4.1

$$\begin{aligned}&\left(\delta_\tau\left(\frac{q(t^{n+1})}{\sqrt{E_2(\mathbf{u}(t^{n+1}))+C_2}}\left((\mathbf{u}(t^{n+1})\cdot\nabla)\mathbf{u}(t^{n+1})\right)\right.\right.\\&\qquad\left.\left.-\frac{q^{n+1}}{\sqrt{E_2(\mathbf{u}^n)+C_2}}\mathbf{u}^n\cdot\nabla\mathbf{u}^n\right),\delta_\tau\tilde{e}_{\mathbf{u}}^{n+1}\right)\\&\leq\frac{\nu}{6}\big\|\nabla\delta_\tau\tilde{e}_{\mathbf{u}}^{n+1}\big\|^2+C\big\|\delta_\tau e_{\mathbf{u}}^n\big\|^2+C\big\|e_{\mathbf{u}}^n\big\|^2+C\big\|e_{\mathbf{u}}^{n-1}\big\|^2\\&\quad+C\big\|e_{\mathbf{u}}^n\big\|_1^2+C\big|\delta_\tau e_q^{n+1}\big|^2+C\big|e_q^{n+1}\big|^2+C\tau^2,\end{aligned}\tag{4.63}$$

where we used the fact that

$$\big\|\nabla\delta_\tau\tilde{e}_{\mathbf{u}}^n\big\|^2\leq\frac{1}{\tau^2}\big\|\nabla\tilde{e}_{\mathbf{u}}^n\big\|^2\leq\frac{C}{\tau},\quad\forall\, 1\leq n\leq N.\tag{4.64}$$

Next the fourth term on the right-hand side of (4.56) can be recast into

$$\begin{aligned}&\left(\delta_\tau\left(\frac{r^{n+1}}{\sqrt{E_1(\phi^n)+C_1}}\mu^n\nabla\phi^n-\frac{r(t^{n+1})}{\sqrt{E_1(\phi(t^{n+1}))+C_1}}\mu(t^{n+1})\nabla\phi(t^{n+1})\right),\delta_\tau\tilde{e}_{\mathbf{u}}^{n+1}\right)\\=&\left(\delta_\tau\left(\frac{r^{n+1}}{\sqrt{E_1(\phi^n)+C_1}}\left(\mu^n\nabla\phi^n-\mu(t^{n+1})\nabla\phi(t^{n+1})\right)\right),\delta_\tau\tilde{e}_{\mathbf{u}}^{n+1}\right)\\&+\left(\delta_\tau\left(\frac{r^{n+1}}{\sqrt{E_1(\phi^n)+C_1}}-\frac{r(t^{n+1})}{\sqrt{E_1(\phi(t^{n+1}))+C_1}}\right)\right.\\&\qquad\qquad\left.\times\mu(t^{n+1})\nabla\phi(t^{n+1}),\delta_\tau\tilde{e}_{\mathbf{u}}^{n+1}\right).\end{aligned}\tag{4.65}$$

Thus, we bounded the (4.65) as follows:

$$\begin{aligned}&\left(\delta_\tau\left(\frac{r^{n+1}}{\sqrt{E_1(\phi^n)+C_1}}\left(\mu^n\nabla\phi^n-\mu(t^{n+1})\nabla\phi(t^{n+1})\right)\right),\delta_\tau\tilde{e}_{\mathbf{u}}^{n+1}\right)\\&\quad+\left(\delta_\tau\left(\frac{r^{n+1}}{\sqrt{E_1(\phi^n)+C_1}}-\frac{r(t^{n+1})}{\sqrt{E_1(\phi(t^{n+1}))+C_1}}\right)\mu(t^{n+1})\nabla\phi(t^{n+1}),\delta_\tau\tilde{e}_{\mathbf{u}}^{n+1}\right)\\&\leq\frac{\nu}{6}\big\|\nabla\delta_\tau\tilde{e}_{\mathbf{u}}^{n+1}\big\|^2+C\big\|\delta_\tau e_\mu^n\big\|^2+C\big\|\delta_\tau\Delta e_\phi^n\big\|^2+C\big\|\delta_\tau\nabla e_\phi^n\big\|^2\\&\quad+C\big\|e_\phi^n\big\|^2+C\big\|e_\phi^{n-1}\big\|^2+C\big|\delta_\tau e_r^{n+1}\big|^2+C\big|e_r^n\big|^2+C\tau^2.\end{aligned}\tag{4.66}$$

Combining (4.56) with (4.57)-(4.66) gets

$$\frac{\|\delta_\tau e_{\mathbf{u}}^{n+1}\|^2-\|\delta_\tau e_{\mathbf{u}}^n\|^2}{2\tau}+\frac{\|\delta_\tau e_{\mathbf{u}}^{n+1}-\delta_\tau e_{\mathbf{u}}^n\|^2}{2\tau}+\frac{\nu}{2}\|\nabla\delta_\tau e_{\mathbf{u}}^{n+1}\|^2$$

$$
\begin{aligned}
&+\frac{\tau}{2}\left(\left\|\nabla\delta_\tau e_p^{n+1}\right\|^2-\left\|\nabla\delta_\tau e_p^n\right\|^2\right)\\
\le{}& \tau^2\left\|\nabla\delta_\tau e_p^n\right\|^2+C\left\|\delta_\tau e_{\mathbf{u}}^n\right\|^2+C\left\|e_{\mathbf{u}}^{n-1}\right\|^2+C\left\|e_{\mathbf{u}}^n\right\|_1^2+C\left\|\delta_\tau\nabla e_\phi^n\right\|^2\\
&+C\left|\delta_\tau e_q^{n+1}\right|^2+C\left|e_q^{n+1}\right|^2+C\left\|\delta_\tau e_\mu^n\right\|^2+C\left\|\delta_\tau\Delta e_\phi^n\right\|^2\\
&+C\left|\delta_\tau e_r^{n+1}\right|^2+C\left|e_r^n\right|^2+C\left\|e_\phi^n\right\|^2+C\left\|e_\phi^{n-1}\right\|^2+C\tau^2. \qquad (4.67)
\end{aligned}
$$

Multiplying (4.67) by 2τ, summing up for n from 1 to m, and using Lemmas 4.5 and 4.6, we obtain

$$
\begin{aligned}
&\left\|\delta_\tau e_{\mathbf{u}}^{m+1}\right\|^2+\nu\tau\sum_{n=1}^m\left\|\nabla\delta_\tau\tilde{e}_{\mathbf{u}}^{n+1}\right\|^2+\tau^2\left\|\nabla\delta_\tau e_p^{m+1}\right\|^2\\
\le{}&\left\|\delta_\tau e_{\mathbf{u}}^1\right\|^2+\tau^2\left\|\nabla\delta_\tau e_p^1\right\|^2+\tau^3\sum_{n=1}^m\left\|\nabla\delta_\tau e_p^n\right\|^2+C\tau\sum_{n=1}^m\left\|\delta_\tau e_{\mathbf{u}}^n\right\|^2\\
&+C\tau\sum_{n=1}^m\left|\delta_\tau e_q^{n+1}\right|^2+C\tau\sum_{n=1}^m\left\|\delta_\tau\nabla e_\phi^n\right\|^2+C\tau\sum_{n=1}^m\left\|\delta_\tau e_\mu^n\right\|^2\\
&+C\tau\sum_{n=1}^m\left\|\delta_\tau e_\phi^n\right\|^2+C\tau\sum_{n=1}^m\left|\delta_\tau e_r^{n+1}\right|^2+C\tau^2\\
\le{}&\left\|\delta_\tau e_{\mathbf{u}}^1\right\|^2+\tau^2\left\|\nabla\delta_\tau e_p^1\right\|^2+\tau^3\sum_{n=1}^m\left\|\nabla\delta_\tau e_p^n\right\|^2+C\tau\sum_{n=1}^m\left\|\delta_\tau e_{\mathbf{u}}^n\right\|^2+C\tau^2. \qquad (4.68)
\end{aligned}
$$

Next we estimate the $\|\delta_\tau e_{\mathbf{u}}^1\|^2+\tau^2\|\nabla\delta_\tau e_p^1\|^2$ terms on the right-hand side of (4.68). Recalling (4.9), we can get

$$
\begin{aligned}
\tilde{e}_{\mathbf{u}}^1-\nu\tau\tilde{e}_{\mathbf{u}}^1={}&\tau\frac{q(t^1)}{\sqrt{E_2(\mathbf{u}(t^1))+C_2}}\left((\mathbf{u}(t^1)\cdot\nabla)\mathbf{u}(t^1)\right) \qquad (4.69)\\
&-\tau\frac{q^1}{\sqrt{E_2(\mathbf{u}^0)+C_2}}\mathbf{u}^0\cdot\nabla\mathbf{u}^0-\tau\nabla\left(p^0-p(t^1)\right)\\
&+\tau\frac{r^1}{\sqrt{E_1(\phi^0)+C_1}}\mu^0\nabla\phi^0-\tau\frac{r(t^1)}{\sqrt{E_1(\phi(t^1))+C_1}}\mu(t^1)\nabla\phi(t^1)+\tau R_{\mathbf{u}}^1.
\end{aligned}
$$

Taking the inner product of (4.69) with $\tilde{e}_{\mathbf{u}}^1$ leads to

$$
\begin{aligned}
&\left\|\tilde{e}_{\mathbf{u}}^1\right\|^2+\nu\tau\left\|\nabla\tilde{e}_{\mathbf{u}}^1\right\|^2\\
={}&\tau\left(\frac{q(t^1)}{\sqrt{E_2(\mathbf{u}(t^1))+C_2}}\left((\mathbf{u}(t^1)\cdot\nabla)\mathbf{u}(t^1)\right)-\frac{q^1}{\sqrt{E_2(\mathbf{u}^0)+C_2}}\mathbf{u}^0\cdot\nabla\mathbf{u}^0,\tilde{e}_{\mathbf{u}}^1\right)\\
&+\tau\left(\frac{r^1}{\sqrt{E_1(\phi^0)+C_1}}\mu^0\nabla\phi^0-\frac{r(t^1)}{\sqrt{E_1(\phi(t^1))+C_1}}\mu(t^1)\nabla\phi(t^1),\tilde{e}_{\mathbf{u}}^1\right)\\
&-\tau\left(\nabla(p^0-p(t^1)),\tilde{e}_{\mathbf{u}}^1\right)+\tau\left(R_{\mathbf{u}}^1,\tilde{e}_{\mathbf{u}}^1\right)\\
\le{}&\frac{1}{2}\left\|\tilde{e}_{\mathbf{u}}^1\right\|^2+C\tau^4. \qquad (4.70)
\end{aligned}
$$

Hence, we have

$$\left\|\delta_\tau e_{\mathbf{u}}^1\right\|^2 \leq \left\|\delta_\tau \tilde{e}_{\mathbf{u}}^1\right\|^2 = \tau^2 \left\|\tilde{e}_{\mathbf{u}}^1\right\|^2 \leq C\tau^2. \tag{4.71}$$

Using (4.11) with $n=1$ results in

$$\tau^2 \left\|\nabla \delta_\tau e_p^1\right\|^2 \leq \frac{1}{\tau^2}\left(\left\|e_{\mathbf{u}}^1\right\|^2 + \left\|\tilde{e}_{\mathbf{u}}^1\right\|^2\right) + \tau^2 \left\|\nabla \delta_\tau p\left(t^1\right)\right\|^2 \leq C\tau^2. \tag{4.72}$$

Substituting the above estimates into (4.68) and applying the discrete Grönwall lemma 2.1, we finally obtain

$$\left\|\delta_\tau e_{\mathbf{u}}^{m+1}\right\|^2 + \nu\tau \sum_{n=1}^{m} \left\|\nabla \delta_\tau \tilde{e}_{\mathbf{u}}^{n+1}\right\|^2 + \tau^2 \left\|\nabla \delta_\tau e_p^{m+1}\right\|^2 \leq C\tau^2. \tag{4.73}$$

The proof is complete. $\square$

Next, we apply Lemmas 4.5-4.7 to prove Theorem 4.2.

Proof. We now give the pressure estimates $(\nabla e_p^{n+1}, \mathbf{v})$ in $\mathbf{v} \in H_0^1(\Omega)$. Taking the inner product of (4.54) with $\mathbf{v}$, we can obtain

$$\begin{aligned}
&\left(\nabla e_p^{n+1}, \mathbf{v}\right) \\
=& -\left(\frac{e_{\mathbf{u}}^{n+1} - e_{\mathbf{u}}^n}{\tau}, \mathbf{v}\right) + \nu\left(\Delta \tilde{e}_{\mathbf{u}}^{n+1}, \mathbf{v}\right) + \left(R_{\mathbf{u}}^{n+1}, \mathbf{v}\right) \\
&+ \left(\frac{q(t^{n+1})}{\sqrt{E_2(\mathbf{u}(t^{n+1})) + C_2}}\left((\mathbf{u}(t^{n+1}) \cdot \nabla)\mathbf{u}(t^{n+1})\right) - \frac{q^{n+1}}{\sqrt{E_2(\mathbf{u}^n) + C_2}}\mathbf{u}^n \cdot \nabla \mathbf{u}^n, \mathbf{v}\right) \\
&+ \left(\frac{r^{n+1}}{\sqrt{E_1(\phi^n) + C_1}}\mu^n \nabla \phi^n - \frac{r(t^{n+1})}{\sqrt{E_1(\phi(t^{n+1})) + C_1}}\mu(t^{n+1})\nabla\phi(t^{n+1}), \mathbf{v}\right).
\end{aligned} \tag{4.74}$$

For (4.74), we can obtain

$$\begin{aligned}
&\left(\frac{q(t^{n+1})}{\sqrt{E_2(\mathbf{u}(t^{n+1})) + C_2}}\left((\mathbf{u}(t^{n+1}) \cdot \nabla)\mathbf{u}(t^{n+1})\right) - \frac{q^{n+1}}{\sqrt{E_2(\mathbf{u}^n) + C_2}}\mathbf{u}^n \cdot \nabla \mathbf{u}^n, \mathbf{v}\right) \\
=& \frac{q(t^{n+1})}{\sqrt{E_2(\mathbf{u}(t^{n+1})) + C_2}}\left((\mathbf{u}(t^{n+1}) - \mathbf{u}^n) \cdot \nabla \mathbf{u}(t^{n+1}), \mathbf{v}\right) \\
&+ \frac{q(t^{n+1})}{\sqrt{E_2(\mathbf{u}(t^{n+1})) + C_2}}\left(\mathbf{u}^n \cdot \nabla(\mathbf{u}(t^{n+1}) - \mathbf{u}^n), \mathbf{v}\right) \\
&+ \left(\frac{q(t^{n+1})}{\sqrt{E_2(\mathbf{u}(t^{n+1})) + C_2}} - \frac{q(t^{n+1})}{\sqrt{E_2(\mathbf{u}^n) + C_2}}\right)\left(\mathbf{u}^n \cdot \nabla \mathbf{u}^n, \mathbf{v}\right) \\
&- \left(\frac{e_q^{n+1}}{\sqrt{E_2(\mathbf{u}^n) + C_2}}\mathbf{u}^n \cdot \nabla \mathbf{u}^n, \mathbf{v}\right) \\
\leq& C\left(\left\|e_{\mathbf{u}}^n\right\| + \left\|\nabla \tilde{e}_{\mathbf{u}}^n\right\| + \left|e_q^{n+1}\right| + \tau\|\mathbf{u}\|_{W^{1,\infty}(0,T;H^1(\Omega))}\right)\|\nabla \mathbf{v}\|.
\end{aligned} \tag{4.75}$$

By using the similar procedure in (4.19), the last term on the right-hand side of (4.74) can be transformed into

$$
\begin{aligned}
&\left(\frac{r^{n+1}}{\sqrt{E_1\left(\phi^n\right)+C_1}}\mu^n\nabla\phi^n-\frac{r\left(t^{n+1}\right)}{\sqrt{E_1\left(\phi\left(t^{n+1}\right)\right)+C_1}}\mu\left(t^{n+1}\right)\nabla\phi\left(t^{n+1}\right),\mathbf{v}\right)\\
=&\left(\frac{r^{n+1}}{\sqrt{E_1(\phi^n)+C_1}}\left(\mu^n\nabla\phi^n-\mu(t^{n+1})\nabla\phi(t^{n+1})\right),\mathbf{v}\right)\\
&+\left(\left(\frac{r^{n+1}}{\sqrt{E_1\left(\phi^n\right)+C_1}}-\frac{r\left(t^{n+1}\right)}{\sqrt{E_1\left(\phi\left(t^{n+1}\right)\right)+C_1}}\right)\mu\left(t^{n+1}\right)\nabla\phi\left(t^{n+1}\right),\mathbf{v}\right)\\
\leq& C\left(\left\|e_\mu^n\right\|+\left\|\nabla e_\mu^n\right\|+\left\|\nabla e_\phi^n\right\|+\left\|e_\phi^n\right\|+\left|e_r^{n+1}\right|+|r|_{W^{1,\infty}(0,T)}\right)\|\nabla\mathbf{v}\|\\
&+C\tau\left(\|\mu\|_{W^{1,\infty}(0,T;H^1(\Omega))}+\|\phi\|_{W^{1,\infty}(0,T;H^1(\Omega))}\right)\|\nabla\mathbf{v}\|.
\end{aligned}\tag{4.76}
$$

Thus, by using the above estimates and the fact that

$$
\left\|e_p^{n+1}\right\|_{L^2(\Omega)/\mathbb{R}}\leq\sup_{\mathbf{v}\in\mathbf{H}_0^1(\Omega)}\frac{(\nabla e_p^{n+1},\mathbf{v})}{\|\nabla\mathbf{v}\|},\tag{4.77}
$$

we finally derive that

$$
\begin{aligned}
&\tau\sum_{n=0}^{m}\left\|e_p^{n+1}\right\|^2_{L^2(\Omega)/\mathbb{R}}\\
\leq& C\tau\sum_{n=0}^{m}\Big(\left\|\delta_\tau e_{\mathbf{u}}^{n+1}\right\|^2+\left\|\nabla\tilde{e}_{\mathbf{u}}^{n+1}\right\|^2+\left\|e_{\mathbf{u}}^{n}\right\|^2+\left|e_q^{n+1}\right|^2\\
&\qquad+\left\|e_\mu^n\right\|^2+\left\|\nabla e_\mu^n\right\|^2+\left\|\nabla e_\phi^n\right\|^2+\left\|e_\phi^n\right\|^2+\left|e_r^{n+1}\right|^2\Big)\\
&+C\tau^2\left(\|\mathbf{u}\|^2_{W^{2,\infty}(0,T;L^2(\Omega))}+\|\mu\|^2_{W^{1,\infty}(0,T;H^1(\Omega))}\right)\\
&+C\tau^2\|\phi\|^2_{W^{1,\infty}(0,T;H^1(\Omega))}.
\end{aligned}\tag{4.78}
$$

The proof is complete. ☐

5. Numerical experiments

In this section, three numerical experiments are performed for the corresponding purposes:

1. Validating convergence rates in Theorems 4.1 and 4.2.
2. Simulating the coarsening dynamics with a random, initial phase function.
3. Showing numerically the unconditional stability of our scheme.

In all tests, P^1 element is chosen for phase function ϕ_h^n and chemical potential μ_h^n, P^2 for $\tilde{\mathbf{u}}_h^n$, and the inf-sup stable pair (P^2,P^1) for $(\mathbf{u}_h^n,p_h^n)$.

5.1. Convergence tests

The first test is performed on $\Omega = [0,1] \times [0,1]$ and time $T = 0.1$, and the physical parameters are selected as $M = 0.1$, $\lambda = 0.001$, $\epsilon = 0.1$ and $\nu = 0.01$, two constants in SAVs are $C_1 = 0$ and $C_2 = 0.1$, and $\gamma = 1$. The time-step τ is set up as $\tau = 0.1h^2$ in the first-order scheme and $\tau = 0.05h$ in the second-order scheme to balance the convergence rates between time and space. The initial conditions are

$$\begin{aligned} \phi^0(x,y) &= \cos(\pi x)\cos(\pi y), \\ \mu^0(x,y) &= -\lambda\Delta\phi^0 + \lambda G'(\phi^0) \\ &= 2\lambda\pi^2\cos(\pi x)\cos(\pi y) + 1/\epsilon^2\lambda\cos(\pi x)\cos(\pi y) \\ &\quad \times \left(\cos^2(\pi x)^2\cos(\pi y) - 1\right), \\ \boldsymbol{u}^0(x,y) &= [\sin(\pi x)^2\sin(2\pi y), -\sin(\pi y)^2\sin(2\pi x)]^{\mathrm{T}}, \\ p^0(x,y) &= 0. \end{aligned} \tag{5.1}$$

We measure the Cauchy error since we do not have an exact solution. Specifically, the errors we computed are defined as the errors between two successive grids with spatial length h and $h/2$, i.e., $\|e_\xi\|_* := \|\xi_h - \xi_{h/2}\|_*$ where ξ denotes some variable and $\|\cdot\|_*$ is the L^2-norm or H^1-norm. Considering the previous settings of τ, we have the following theoretical convergence rates:

$$\begin{aligned} &\left\|\phi^{n+1} - \phi_h^{n+1}\right\|_{L^2} \approx \mathcal{O}(h^2), \quad \left\|\phi^{n+1} - \phi_h^{n+1}\right\|_{H^1} \approx \mathcal{O}(h), \\ &\left\|\mu^{n+1} - \mu_h^{n+1}\right\|_{L^2} \approx \mathcal{O}(h^2), \quad \left\|\mu^{n+1} - \mu_h^{n+1}\right\|_{H^1} \approx \mathcal{O}(h), \\ &\left\|\boldsymbol{u}^{n+1} - \boldsymbol{u}_h^{n+1}\right\|_{L^2} \approx \mathcal{O}(h^2), \quad \left\|\boldsymbol{u}^{n+1} - \boldsymbol{u}_h^{n+1}\right\|_{H^1} \approx \mathcal{O}(h^2), \\ &\left\|p^{n+1} - p_h^{n+1}\right\|_{L^2} \approx \mathcal{O}(h^2), \quad \left\|p^{n+1} - p_h^{n+1}\right\|_{H^1} \approx \mathcal{O}(h). \end{aligned} \tag{5.2}$$

The errors and convergence rates illustrated in Tables 1 and 2 for first-order scheme and in Tables 3 and 4 for second-order scheme are consistent with the above a prior rates.

Table 1: L^2 Errors and convergence orders for first-order scheme.

h	$h/2$	$\|\phi^N - \phi_h^N\|_{L^2}$		$\|\mu^N - \mu_h^N\|_{L^2}$		$\|\boldsymbol{u}^N - \boldsymbol{u}_h^N\|_{L^2}$		$\|p^N - p_h^N\|_{L^2}$	
		error	rate	error	rate	error	rate	error	rate
$1/4$	$1/8$	5.0819e−02	-	7.1315e−03	-	4.4180e−02	-	5.1038e−02	-
$1/8$	$1/16$	1.5372e−02	1.73	2.4467e−03	1.54	9.9959e−03	2.14	1.2065e−02	2.08
$1/16$	$1/32$	3.8458e−03	2.00	6.9875e−04	1.81	6.6334e−04	3.91	2.5614e−03	2.24
$1/32$	$1/64$	9.5963e−04	2.00	1.7946e−04	1.96	5.7945e−05	3.52	6.1613e−04	2.06

Table 2: H^1 Errors and convergence orders for first-order scheme.

h	$h/2$	$\|\phi^N - \phi_h^N\|_{H^1}$		$\|\mu^N - \mu_h^N\|_{H^1}$		$\|\boldsymbol{u}^N - \boldsymbol{u}_h^N\|_{H^1}$		$\|p^N - p_h^N\|_{H^1}$	
		error	rate	error	rate	error	rate	error	rate
$1/4$	$1/8$	8.3731e−01	-	1.2795e−01	-	9.7682e−01	-	9.7374e−01	-
$1/8$	$1/16$	5.0831e−01	0.72	7.0683e−02	0.86	4.5073e−01	1.12	6.0342e−01	0.69
$1/16$	$1/32$	2.6015e−01	0.97	3.6548e−02	0.95	7.3778e−02	2.61	2.8647e−01	1.07
$1/32$	$1/64$	1.3037e−02	1.00	1.8357e−02	0.99	1.3226e−02	2.48	1.4118e−01	1.02

Table 3: L^2 Errors and convergence orders for second-order scheme.

h	$h/2$	$\|\phi^N - \phi_h^N\|_{L^2}$		$\|\mu^N - \mu_h^N\|_{L^2}$		$\|\boldsymbol{u}^N - \boldsymbol{u}_h^N\|_{L^2}$		$\|p^N - p_h^N\|_{L^2}$	
		error	rate	error	rate	error	rate	error	rate
1/4	1/8	4.8444e−02	-	6.4454e−03	-	2.6722e−02	-	5.5861e−02	-
1/8	1/16	1.3211e−02	1.87	1.7992e−03	1.84	7.9763e−03	1.73	1.3001e−02	2.10
1/16	1/32	3.4479e−03	1.94	4.9464e−04	1.86	7.1216e−04	3.49	2.7217e−03	2.26
1/32	1/64	9.4911e−04	1.86	1.2306e−04	2.00	1.1908e−04	2.58	6.6224e−04	2.04

Table 4: H^1 Errors and convergence orders for second-order scheme.

h	$h/2$	$\|\phi^N - \phi_h^N\|_{H^1}$		$\|\mu^N - \mu_h^N\|_{H^1}$		$\|\boldsymbol{u}^N - \boldsymbol{u}_h^N\|_{H^1}$		$\|p^N - p_h^N\|_{H^1}$	
		error	rate	error	rate	error	rate	error	rate
1/4	1/8	7.6376e−01	-	1.0661e−01	-	6.8707e−01	-	9.9117e−01	-
1/8	1/16	4.1387e−01	0.88	5.3378e−02	1.00	3.6927e−01	0.89	6.2954e−01	0.65
1/16	1/32	2.1091e−01	0.97	2.9851e−02	0.84	7.6916e−02	2.26	3.0385e−01	1.05
1/32	1/64	1.0591e−01	0.99	1.5419e−02	0.95	1.3754e−02	2.48	1.4963e−01	1.02

5.2. Coarsening dynamics

In this example, we choose $\Omega = [0,1] \times [0,1]$, and $M = 0.0001$, $\lambda = 0.02$, $\epsilon = 0.01$, $\nu = 1$, $C_1 = 1$, $C_2 = 0.1$, and $\gamma = 1$, with a random initial condition for the phase function with values in $[-0.1, 0.1]$, the initial chemical potential takes the same random values as phase function. The spatial length $h = 1/64$, and $\tau = 0.001$. We run this test up to final time $T = 5$, and record the snapshots of phase function at $t = 0.001, 0.05, 0.1, 0.15, 0.3, 1, 3, 5$, respectively, in Fig. 1.

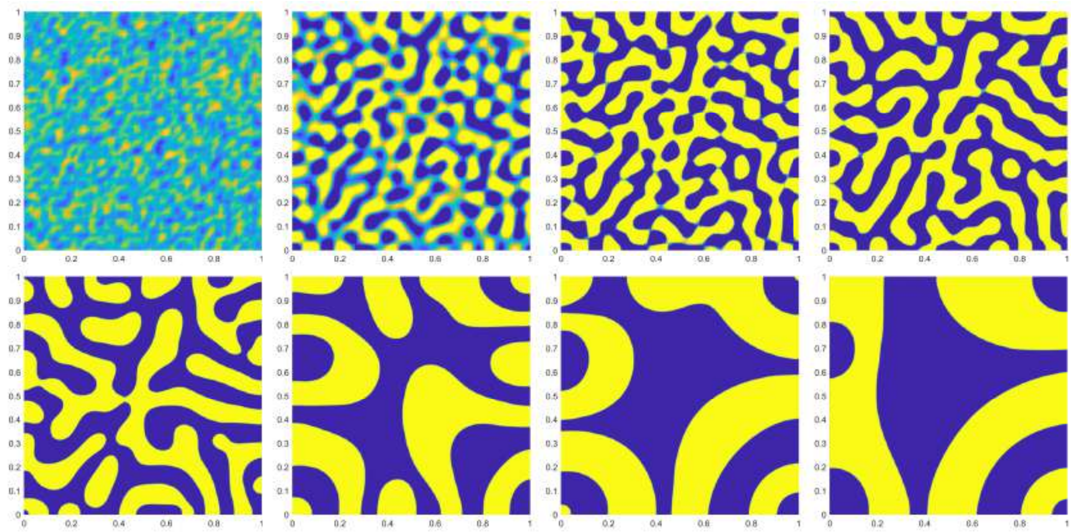

Figure 1: Snapshots of phase function at difference times from left to right row by row with $t = 0.001, 0.05, 0.1, 0.15, 0.3, 1, 3, 5$, respectively.

5.3. Verification of unconditional stability

In this final test, we aim to verify the property of unconditional stability of the new scheme here. The same settings as the one in the above coarsening dynamics are chosen but with varying time-step τ. From Fig. 2 we observe that our scheme is indeed unconditionally stable in the sense of energy dissipation.

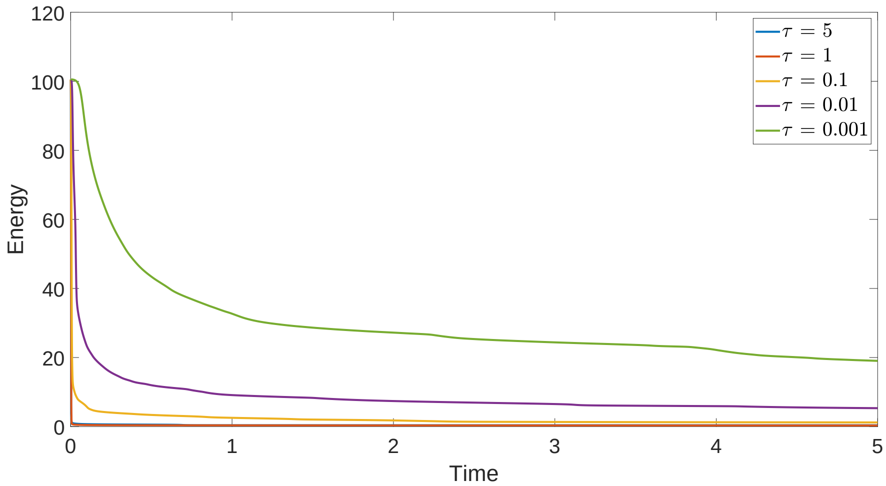

Figure 2: Evolution of the energy with different time-step τ.

6. Conclusions

The CHNS model of two-phase incompressible flows is a highly coupled nonlinear system based on the energy dissipation law. In this article, we constructed an efficient time semi-discretization scheme based on the SAVs methods for the Cahn-Hilliard-Navier-Stokes model of two-phase incompressible flows, which deals with the nonlinear terms of the CHNS and combines with the pressure correction methods or rotational-pressure correction methods to deal with the coupling of pressure and velocity. The nonlinear terms in the equations are treated by the SAV method, and the coupling terms are treated explicitly to prove that our schemes are unconditionally energy stable. We also give error estimates for the first-order time semi-discrete scheme. Finally, numerical experiments show that our discrete format meets the theoretical requirements.

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References

[1] K. BAO, Y. SHI, S. SUN, AND X. WANG, *A finite element method for the numerical solution of the coupled Cahn-Hilliard and Navier-Stokes system for moving contact line problems*, J. Comput. Phys. 231 (2012), 8083–8099.

[2] J. CAHN AND J. HILLIARD, *Free energy of a nonuniform system. I. Interfacial Free Energy*, J. Chem. Phys. 28 (1958), 258-–267.

[3] Y. CAI, H. CHOI, AND J. SHEN, *Error estimates for time discretizations of Cahn-Hilliard and Allen-Cahn phase-field models for two-phase incompressible flows*, Numer. Math. 137 (2017), 417–449.

[4] Y. CAI AND J. SHEN, *Error estimates for a fully discretized scheme to a Cahn-Hilliard phase-field model for two-phase incompressible flows*, Math. Comp. 87 (2018), 2057–2090.

[5] W. CHEN, J. JING, C. WANG, X. WANG, AND S. WISE, *A modified Crank-Nicolson numerical scheme for the Flory-Huggins Cahn-Hilliard model*, Commun. Comput. Phys. 31 (2022), 60–93.

[6] Y. CHEN, Y. HUANG, AND N. YI, *Error analysis of a decoupled, linear and stable finite element method for Cahn-Hilliard-Navier-Stokes equations*, Appl. Math. Comput. 421 (2022), Paper No. 126928.

[7] Y. CHEN AND J. SHEN, *Efficient, adaptive energy stable schemes for the incompressible Cahn-Hilliard Navier-Stokes phase-field models*, J. Comput. Phys. 308 (2016), 40–56.

[8] A. DIEGEL, X. FENG, AND S. WISE, *Analysis of a mixed finite element method for a Cahn-Hilliard-Darcy-Stokes system*, SIAM J. Numer. Anal. 53 (2015), 127–152.

[9] A. DIEGEL, C. WANG, X. WANG, AND S. WISE, *Convergence analysis and error estimates for a second order accurate finite element method for the Cahn-Hilliard-Navier-Stokes system*, Numer. Math. 137 (2017), 495–534.

[10] X. FENG, *Fully discrete finite element approximations of the Navier-Stokes-Cahn-Hilliard diffuse interface model for two-phase fluid flows*, SIAM J. Numer. Anal. 44 (2006), 1049–1072.

[11] J. L. GUERMOND, P. MINEV, AND J. SHEN, *An overview of projection methods for incompressible flows*, Comput. Methods Appl. Mech. Engrg. 195 (2006), 6011–6045.

[12] Z. GUO, P. LIN, AND J. LOWENGRUB, *A numerical method for the quasi-incompressible Cahn-Hilliard-Navier-Stokes equations for variable density flows with a discrete energy law*, J. Comput. Phys. 276 (2014), 486–507.

[13] D. HAN AND X. WANG, *A second order in time, uniquely solvable, unconditionally stable numerical scheme for Cahn-Hilliard-Navier-Stokes equation*, J. Comput. Phys. 290 (2015), 139–156.

[14] Q. HE, R. GLOWINSKI, AND X. WANG, *A least-squares/finite element method for the numerical solution of the Navier-Stokes-Cahn-Hilliard system modeling the motion of the contact line*, J. Comput. Phys. 230 (2011), 4991–5009.

[15] Y. HE AND W. SUN, *Stability and convergence of the Crank-Nicolson/Adams-Bashforth scheme for the time-dependent Navier-Stokes equations*, SIAM J. Numer. Anal. 45 (2007), 837–869.

[16] H. JIA, X. WANG, AND K. LI, *A novel linear, unconditional energy stable scheme for the incompressible Cahn-Hilliard-Navier-Stokes phase-field model*, Comput. Math. Appl. 80 (2020), 2948–2971.

[17] D. KAY, V. STYLES, AND R. WELFORD, *Finite element approximation of a Cahn-Hilliard-Navier-Stokes system*, Interfaces Free Bound. 10 (2008), 15–43.

[18] D. KAY AND R. WELFORD, *Efficient numerical solution of Cahn-Hilliard-Navier-Stokes fluids*

in 2D, SIAM J. Sci. Comput. 29 (2007), 2241–2257.

[19] M. LI AND C. XU, *New efficient time-stepping schemes for the Navier-Stokes-Cahn-Hilliard equations*, Comput. & Fluids 231 (2021), Paper No. 105174.

[20] X. LI AND J. SHEN, *Error analysis of the SAV-MAC scheme for the Navier-Stokes equations*, SIAM J. Numer. Anal. 58 (2020), 2465–2491.

[21] X. LI AND J. SHEN, *On a SAV-MAC scheme for the Cahn-Hilliard-Navier-Stokes phase-field model and its error analysis for the corresponding Cahn-Hilliard-Stokes case*, Math. Models Methods Appl. Sci. 30 (2020), 2263–2297.

[22] X. LI AND J. SHEN, *On fully decoupled MSAV schemes for the Cahn-Hilliard-Navier-Stokes model of two-phase incompressible flows*, Math. Models Methods Appl. Sci. 32 (2022), 457–495.

[23] X. LI, J. SHEN, AND Z. LIU, *New SAV-pressure correction methods for the Navier-Stokes equations: Stability and error analysis*, Math. Comp. 91 (2021), 141–167.

[24] L. LIN, Z. YANG, AND S. DONG, *Numerical approximation of incompressible Navier-Stokes equations based on an auxiliary energy variable*, J. Comput. Phys. 388 (2019), 1–22.

[25] J. H. PARK, A. SALGADO, AND S. WISE, *Benchmark computations of the phase field crystal and functionalized Cahn-Hilliard equations via fully implicit, Nesterov accelerated schemes*, Commun. Comput. Phys. 33 (2023), 367–398.

[26] J. SHEN AND J. XU, *Convergence and error analysis for the scalar auxiliary variable (SAV) schemes to gradient flows*, SIAM J. Numer. Anal. 56 (2018), 2895–2912.

[27] J. SHEN, J. XU, AND J. YANG, *The scalar auxiliary variable (SAV) approach for gradient flows*, J. Comput. Phys. 353 (2018), 407–416.

[28] J. SHEN, J. XU, AND J. YANG, *A new class of efficient and robust energy stable schemes for gradient flows*, SIAM Rev. 61 (2019), 474–506.

[29] J. SHEN AND X. YANG, *Energy stable schemes for Cahn-Hilliard phase-field model of two-phase incompressible flows*, Chinese Ann. Math. Ser. B 31 (2010), 743–758.

[30] R. TEMAM, *Navier-Stokes Equations: Theory and Numerical Analysis*, Studies in Mathematics and its Applications, Vol. 2, North-Holland Publishing Co., 1977.

[31] R. TEMAM, *Navier-Stokes Equations and Nonlinear Functional Analysis*, CBMS-NSF Regional Conference Series in Applied Mathematics, Vol. 66, SIAM, 1995.

[32] L. WANG, Y. HUANG, AND K. JIANG, *Error analysis of SAV finite element method to phase field crystal model*, Numer. Math. Theor. Meth. Appl. 13 (2020), 372–399.

[33] J. YANG, N. YI, AND Y. CHEN, *Optimal error estimates of a SAV-FEM for the Cahn-Hilliard-Navier-Stokes model*, J. Comput. Appl. Math. 438 (2024), Paper No. 115577.

[34] X. YANG, J. ZHAO, AND Q. WANG, *Numerical approximations for the molecular beam epitaxial growth model based on the invariant energy quadratization method*, J. Comput. Phys. 333 (2017), 104–127.

[35] J. ZHAO, Q. WANG, AND X. YANG, *Numerical approximations for a phase field dendritic crystal growth model based on the invariant energy quadratization approach*, Internat. J. Numer. Methods Engrg. 110 (2017), 279–300.