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A Divergence-Free Pk CDG Finite Element for the Stokes Equations on Triangular and Tetrahedral Meshes

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Abstract. In the conforming discontinuous Galerkin method, the standard bilinear form for the conforming finite elements is applied to discontinuous finite elements without adding any inter-element nor penalty form. The P_k $(k \geq 1)$ discontinuous finite elements and the P_{k-1} weak Galerkin finite elements are adopted to approximate the velocity and the pressure respectively, when solving the Stokes equations on triangular or tetrahedral meshes. The discontinuous finite element solutions are divergence-free and surprisingly H-div functions on the whole domain. The optimal order convergence is achieved for both variables and for all $k \geq 1$. The theory is verified by numerical examples.

AMS subject classifications: 65N15, 65N30
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1. Introduction

In this paper, we introduce a divergence-free conforming discontinuous Galerkin method to solve the Stokes equations, finding unknown velocity $\mathbf{u}$ and pressure p such that

$$-\mu\Delta\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega, \tag{1.1}$$

$$\nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega, \tag{1.2}$$

$$\mathbf{u} = 0 \quad \text{on } \partial\Omega, \tag{1.3}$$

where viscosity $\mu > 0$ is a constant, and Ω is a polygonal or polyhedral domain in $\mathbb{R}^d$ $(d = 2, 3)$. The weak form of the Stokes equations is: Find $(\mathbf{u}, p) \in H_0^1(\Omega) \times L_0^2(\Omega)$

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such that

$$(\mu\nabla\mathbf{u},\nabla\mathbf{v})-(p,\nabla\cdot\mathbf{v})=(\mathbf{f},\mathbf{v}),\quad \forall\,\mathbf{v}\in H_0^1(\Omega), \tag{1.4}$$

$$(q,\nabla\cdot\mathbf{u})=0,\qquad \forall\,q\in L_0^2(\Omega), \tag{1.5}$$

where $L_0^2(\Omega)$ is the L^2 space with mean value 0 on the domain.

To get divergence-free finite element solutions, the conforming finite element method employs the continuous P_k finite elements and the discontinuous P_{k-1} finite elements approximating the velocity and the pressure respectively on triangular or tetrahedral meshes, using the bilinear forms in (1.4)-(1.5). However, most such combinations are not stable. The first such a working method was discovered by Scott and Vogelius [34, 35], on 2D triangular meshes for all $k\geq 4$, provided that the underlying meshes have no nearly-singular vertex. Such divergence-free finite elements are studied on special 2D and 3D meshes, or with high-order polynomial and/or macro bubbles, in [3, 7, 8, 10–12, 14–16, 18, 22, 27, 30–33, 44, 60–68, 68].

Another method, to get divergence-free finite element solutions, is to employ $H(\text{div})$ finite element functions on triangular and tetrahedral meshes. That is, the $H(\text{div})$-P_k/discontinuous-P_{k-1} mixed element also produces divergence-free solutions, cf. [26, 37, 39, 42]. However, as the $H(\mathrm{div})$ finite element is not H^1-conforming, in most cases, additional inter-element and penalty forms are added to the variational formulation (1.4)-(1.5). One can get rid of all such stabilizers by using proper weak gradients and weak divergences [28, 29, 49, 50, 69].

A completely new method, to get divergence-free finite element solutions, is to be proposed in this work. We start with totally discontinuous P_k ($k\geq 1$) polynomials to approximate the velocity. The resulting discontinuous P_k solutions are no longer discontinuous, but are continuous in the normal direction on all edges/triangles. That is, the finite element solutions are in the $H(\text{div},\Omega)$ space and are point-wise divergence-free. Let $\mathcal{T}_h=\{T\}$ be a quasi-uniform triangular or tetrahedral mesh with mesh-size h. The conforming discontinuous Galerkin (CDG) finite element spaces are defined by

$$\mathbf{V}_h=\left\{\mathbf{u}_h\in L^2(\Omega):\mathbf{u}_h|_T\in\mathbf{P}_k(T),\ T\in\mathcal{T}_h\right\}, \tag{1.6}$$

where $\mathbf{P}_k(T)$ is the space of 2D or 3D vector polynomials of degree $k\geq 1$ or less on T. Some references on the CDG methods can be found in [9, 47, 48, 51–56]. The weak Galerkin (WG) finite element spaces are defined by

$$\begin{aligned} P_h=\big\{q_h=\{q_0,q_b\}:q_0|_T\in P_{k-1}(T),\ T\in\mathcal{T};\ q_b|_e\in P_k(e),\\ e\in\mathcal{E}_h;(q_0,1)_{\mathcal{T}_h}+\langle q_b,1\rangle_{\partial\mathcal{T}_h}=0\big\}, \end{aligned} \tag{1.7}$$

where $\mathcal{E}_h$ is the set of edges or face-triangles in mesh $\mathcal{T}_h$, $(\cdot,\cdot)_{\mathcal{T}_h}=\sum_{T\in\mathcal{T}_h}(\cdot,\cdot)_T$ and $\langle\cdot,\cdot\rangle_{\partial\mathcal{T}_h}=\sum_{T\in\mathcal{T}_h}\langle\cdot,\cdot\rangle_{\partial T}$. Some references for the WG methods are [1, 2, 4, 6, 13, 17, 19–21, 23–25, 36, 38, 40, 41, 43, 46, 57–59, 70]. The proposed CDG finite element method for the Stokes equations reads: Find $(\mathbf{u}_h,p_h)\in\mathbf{V}_h\times P_h$ such that

$$(\mu\nabla_w\mathbf{u}_h,\nabla_w\mathbf{v}_h)+(\nabla_w p_h,\mathbf{v}_h)=(\mathbf{f},\mathbf{v}_h),\quad \forall\,\mathbf{v}_h\in\mathbf{V}_h, \tag{1.8}$$

$$(\nabla_w q_h,\mathbf{u}_h)=0,\qquad \forall\,q_h\in P_h, \tag{1.9}$$