Numer. Math. Theor. Meth. Appl. Vol. **18**, No. 1, pp. 175-198
doi: 10.4208/nmtma.OA-2024-0076 February 2025

The Direct Discontinuous Galerkin Method with Explicit-Implicit-Null Time Discretizations for Nonlinear Diffusion Equations

Yumiao Li[1], Yin Yang[2,*], Tiegang Liu[3], Weixiong Yuan[3] and Kui Cao[3]

[1] *Hunan International Scientific and Technological Innovation Cooperation Base of Computational Science, Xiangtan University, Xiangtan 411105, China*
[2] *National Center for Applied Mathematics in Hunan, Xiangtan University, Xiangtan 411105, China*
[3] *LMIB and School of Mathematical Sciences, Beihang University, Beijing 100191, China*

Received 20 June 2024; Accepted (in revised version) 31 October 2024

Abstract. This paper proposes a discussion of the direct discontinuous Galerkin (DDG) methods coupled with explicit-implicit-null time discretizations (EIN) for solving the nonlinear diffusion equation $u_t = (a(u)u_x)_x$. The basic idea of the EIN method is to add and subtract two equal constant coefficient terms a_1u_{xx} ($a_1 = a_0 \times \max_u a(u)$) on the right-hand side of the above equation, and then apply the explicit-implicit time-marching method to the equivalent equation. The EIN method does not require any nonlinear iterative solver while eliminating the severe time-step restrictions typically associated with explicit methods. We present the stability criterion of the EIN-DDG schemes for the simplified equation with periodic boundary conditions via the Fourier method, where the first order and second order EIN-DDG schemes are unconditionally stable when $a_0 \geq 0.5$ and the third order EIN-DDG scheme is unconditionally stable under the condition $a_0 \geq 0.54$. Numerical experiments show the stability and optimal orders of accuracy of our proposed schemes with a relaxed time-step restriction and the appropriate coefficient a_0 for both linear and nonlinear equations in one-dimensional and two-dimensional settings. Furthermore, we also show that the computational efficiency of our EIN-DDG schemes and explicit Runge-Kutta DDG (EX-RK-DDG) schemes for steady-state equations with small viscosity coefficients to illustrate the effectiveness of the present methods.

AMS subject classifications: 65M60, 65M12, 65M15

Key words: Direct discontinuous Galerkin method, explicit-implicit-null time discretization, stability, nonlinear diffusion equation, steady-state equation.

*Corresponding author. *Email address:* `yangyinxtu@xtu.edu.cn` (Y. Yang)

1. Introduction

Diffusion is a common phenomenon in nature and has been studied in areas such as percolation, phase change, biochemistry, and population dynamics. It can be effectively modeled using nonlinear diffusion equations. The numerical study of nonlinear diffusion equations has attracted considerable attention from many scholars who are committed to developing higher order numerical methods with stability and convergence.

Although the explicit time-marching method is relatively straightforward to implement, its stability is constrained by the severe time-step $\tau = \mathcal{O}(h^k)$ for the k-th $(k \geq 2)$ order partial differential equations (PDEs), which results in high computational costs and renders the explicit scheme impractical. For example, under a strict CFL-like stability condition $c_0\tau \leq \epsilon \leq c_1 h^2$, Liu and Wen [20] proved the third order explicit Runge-Kutta time discretization with the alternating evolution discontinuous Galerkin scheme is stable for linear convection-diffusion equations. The implicit time-marching method can overcome the limitation of a small time-step and can be applied to any order [14]. However, a fully implicit method is not always optimal for solving nonlinear equations, as it necessitates the resolution of a non-symmetric, non-positively deterministic, and nonlinear algebraic system at each time-step [6, 8, 13, 15, 17]. Jay [16] employed the preconditioned linear iterative method to solve approximately the linear systems of the simplified Newton method. However, the above linear system requires computing and storing the Jacobian of nonlinear operators, and its fast solution relies on an efficient preconditioner, which increases the difficulty of the implicit time-marching method. In order to overcome such difficulties, the implicit-explicit (IMEX) time-marching methods [1, 2, 12, 23, 24] have been proposed and treated the higher order derivative terms implicitly and the rest of the terms explicitly. Such a treatment permits a portion of the solution to be explicit, which is typically more efficient than the fully implicit method. Nevertheless, due to each implicit stage requiring solving a nonlinear system, the method may not apply to equations where both the convection and diffusion terms are nonlinear.

To address the abovementioned issues, Douglas and Dupont [7] proposed and adopted a method to guarantee the stability of nonlinear diffusion equations on a rectangular domain. Later, Duchemin and Eggers [9] proposed and referred to that method as explicit-implicit-null method. We take the one-dimensional nonlinear diffusion equation as an example to illustrate the idea of the EIN method. Adding and subtracting the equal term $a_1 u_{xx}$ on the right-hand side of the equation $u_t = (a(u)u_x)_x$, we obtain

$$u_t = \underbrace{(a(u)u_x)_x - a_1 u_{xx}}_{T_1} + \underbrace{a_1 u_{xx}}_{T_2}, \quad a_1 = a_0 \times \max_u a(u), \tag{1.1}$$

where $a(u) \geq 0$ is bounded and smooth, a_0 is a stabilization parameter and is constant. We treat the term T_1 explicitly and the term T_2 implicitly. Here, the EIN method does not require any nonlinear iterative solver while eliminating the typically severe time-step restrictions, which combines the advantages of both explicit and implicit methods. Recently, the EIN methods coupled with spatial discretizations were success-